Victor Rocha da Silva

Perturbative Re-Construction of the Massive Derivative Coupling Model in 2 Dimensions

Juiz de Fora - MG, Brasil

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Dissertação apresentada ao Programa de Pós-Graduação em Física do Instítuto de Ciências Exatas da Universidade Federal de Juiz de Fora como requisito parcial a obtenção do título de Mestre em Física.

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Resumo

O modelo de Schroer [1] é um modelo bidimensional construído a fim de discutir as características estruturais da Eletrodinâmica Quântica, mais especificamente, as particularidades que ocorrem no espaço de Hilbert e na dinâmica devido ao caráter de infrapartícula do elétron. A interação nesse modelo é dada pelo bóson sem massa ϕ. Os campos do modelo não vivem no espaço de Fock de férmions livres devido a divergências no infravermelho e, portanto, é necessário definir o modelo através das funções de Wightman e reconstruir o espaço de Hilbert. Neste trabalho estudamos o modelo que contêm o bóson ϕ de massa m que é livre, no sentido de obedecer a equação de Klein-Gordon, e um férmion ψ_q de massa M, que são acoplados pela equação de movimento $(i\partial\!\!\!/ - M)\psi_q = -q(\partial\!\!\!/\phi)\psi_q$, onde q é a constante de acoplamento. A solução não-perturbativa é dada pelo campo de Dirac livre vestido $\psi_q \doteq : e^{iq\phi(x)}\psi(x) :$ de [1], onde ψ é o campo de Dirac livre. Nós o chamaremos de modelo de Schroer massivo. As divergências no infravermelho não aparecem no caso massivo. Aqui, sugerimos como o modelo de Schroer massivo surge a partir do campo de Dirac livre com a interação $\mathcal{L}_{int} = \partial_\mu\phi j^\mu$ no contexto da teoria de perturbação de Epstein-Glaser, com ϕ sendo o bóson massivo e j^μ a corrente de Dirac. Esse modelo é renormalizável, com um número infinito de gráficos a serem normalizados. Nós então impomos certas condições de normalização, que entre outras, estão as identidades de Ward extendidas. Para gráficos de árvore, essas condições de normalização são automaticamente satisfeitas, enquanto gráficos com loops são fixados unicamente pelas respectivas normalizações. Isso torna o modelo superrenormalizável. Nós mostramos que, no limite adiabático, a matrix S é igual a unidade, os observáveis interagentes $j^\mu, \partial_\mu\phi$ se tornam livres e a versão interativa do campo de Dirac livre coincide com o campo de Dirac livre vestido ψ_q mencionado acima.

Palavras-chave: Campos quânticos. Eletrodinâmica Quântica. Modelo de Schroer. Epstein-Glaser. Renormalização

Abstract

The Schroer model [1] is a 2-dimensional model built to discuss the structural characteristics of Quantum Electrodynamics (QED), namely the Hilbert space and dynamic particularities due to the infraparticle character of the electron. The interaction there is set through a massless boson ϕ. In this case, the fields do not live in the Fock space of free fermions due to IR divergences, and so one has to define the model through the Wightman functions and then reconstruct the Hilbert space. The model we studied contains the boson ϕ of mass m that is free, in the sense of obeying Klein-Gordon equation, and a fermion ψ_q of mass M, which are coupled through the equation of motion $(i\partial\!\!\!/ - M)\psi_q = -q(\partial\!\!\!/\phi)\psi_q$, where q is the coupling constant. The non-perturbative solution is the dressed Dirac field $\psi_q(x) \doteq: e^{iq\phi(x)} : \psi(x)$ from [1], where ϕ is the free boson and ψ is the free Dirac field. We will call this the massive Schroer model. The IR divergences do not appear in the massive case. We suggest how the massive Schroer model arise from the free Dirac field with the interaction $\mathcal{L} = \partial_\mu\phi j^\mu$ in the context of Epstein-Glaser perturbation theory, with ϕ being the massive boson and j^μ the Dirac current. This model is renormalizable, with an infinite number of graphs to be normalized. We impose certain normalization conditions, which among others are the extended Ward identities. For tree graphs, these normalization conditions are automatically satisfied, while loop graphs are uniquely fixed by the respective normalization. This turns the model superrenormalizable. We show that, in the adiabatic limit, the S-matrix equals the unity, the interacting observables $j^\mu, \partial_\mu\phi$ become free fields, and the interacting version of the free Dirac field coincides with the free dressed Dirac field ψ_q mentioned above.

Keywords: Quantum Fields. Quantum Electrodynamics. Schroer Model. Epstein-Glaser. Renormalization.

Notation

γ^μ Dirac's gamma matrix

In two dimensions, the two gamma matrices can be given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

ψ Dirac fermion

ψ^* Hermitian conjugate

$\psi^\dagger = (\psi^*)^T$

$\overline{\psi} \doteq \psi^\dagger \gamma^0$

$j^\mu =: \overline{\psi}\gamma^\mu\psi :$ Dirac current

$\not{\partial} \doteq \gamma^\mu \partial_\mu$

$: e^{if} : = \sum_{k=0}^{\infty} \frac{i^k}{k!} : f^k :$

$\eta^{\mu\nu}$ Minkowski metric $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = diag(1, -1)$

Throughout this work we shall use natural units, i.e., $c = \hbar = 1$.

Summary

Introduction

The most accurate theory in history is Quantum Electrodynamics (QED), agreeing to twelve decimal places with experiments [2]. Nevertheless, we still struggle with the Infrared (IR) divergences that naturally occur in the structure of QED. These divergences were initially found by looking at the scattering of electrons by atomic nuclei. Mott was the first to address these divergences in [3, 4]. Bloch and Nordsieck also came across this infrared problem and presented a model describing an ideal scattering process as a way to contour the divergences [5]. They showed that the probability of a finite number of "soft" photons[1] escaping detection is zero. In other words, in a scattering of a charged particle the emitted radiation has a finite energy but an infinite number of soft photons. The main result of the Bloch-Nordsieck model is that the cross-section obtained is finite. More than two decades later, Yennie, Frautschi and Suura [6] built a treatment of the infrared divergences based on Bloch-Nordsieck and Pauli-Fierz's [7] models. These developments motivated the first ideas about infraparticles by Bert Schroer [1] in 1963[2,3].

Buchholz introduced in 1982 a characterization of infraparticles and showed the existence of asymptotic electromagnetic fields in all charge sectors [10]. In 1986, he showed that through a quantum version of the Gauss Law [11] an electrically charged state cannot be an eigenstate of the mass operator, obtaining thereby the infraparticle structure and that a spontaneous breakdown of the Lorentz symmetry happens in charged supersectors.

In 1974, Ferrari, Picasso and Strocchi proved that electrically charged fields cannot be pointlike localized [12]. Buchholz and Fredenhagen [13] showed later in 1982, that the most general localization allowed by the mass gap hypothesis combined with the existence of a pointlike generated neutral subalgebra is a semi-infinite spacelike cone. Following this idea, Mund, Yngvason and Schroer [14, 15] in the '00s presented the string-localized quantum fields, where the "strings" are idealized narrow spacelike cones. This type of localization can improve the UV behaviour of perturbative interactions and avoids the use of an indefinite metric [15–17].

In the last decade infraparticles came back to the attention of the community as the problem of infrared divergences resurfaced [18–28]. We will mention a few recent important papers. Mund, Schroer and Rehren [29] addressed the relation between the implications of the Gauss Law and the structure obtained in [1] considering string-localized fields. Dybalski and Mund also computed a scattering amplitude for charged infraparticles living in the GNS representation of the two dimensional massless scalar free field [30]. Mund, Rehren

[1] Photons with energy less than the energy resolution of the experimental apparatus.
[2] For a complete historical background of the approaches made to avoid the infrared divergences see [8].
[3] An anthology on infraparticles was made by Prof. Schroer in [9].

and Schroer [31] presented a new roadmap towards the perturbative off-shell construction of QED, including its charged fields, with Hilbert space positivity being the guidance. Their construction is based in string-localized quantum fields and helps to clarify some parts of the so-called "infrared triangle" composed of the relations between soft photon theorems, asymptotic symmetries and memory effects[4].

The purpose of this work is to implement the perturbative construction of a model containing a free boson ϕ of mass m and a fermion ψ_q of mass M, which are coupled through the equation of motion $(i\partial\!\!\!/ - M)\psi_q = -q\partial\!\!\!/\phi\psi_q$, where q is the coupling constant. The non-perturbative solution is the dressed Dirac field $\psi_q(x) \doteq: e^{iq\phi(x)} : \psi(x)$ from [1], where ϕ is the free boson and ψ is the free Dirac field. The interaction of this model is $\partial_\mu \phi j^\mu$, where ϕ is the massive boson and j^μ is the Dirac current. We call this model the massive Schroer model. Our aim was to show that the perturbative construction coincide with the exact solution of the model. Since we consider a massive boson, the infrared divergence will not appear.

This master thesis is composed of three chapters. The first chapter contains key basic concepts. We present some important mathematical topics, a brief review on the structure of spacetime and the Wightman axiomatic construction of fields. The second chapter comprises a general set-up of the Epstein-Glaser scheme. Our motivation to work within this scheme, also known as causal perturbation theory, is that no ultraviolet divergences appear, i.e. the time-ordered products are finite and well defined. The only adversity we face is a non-uniqueness of the time-ordered products due to finite renormalization terms. In the first section of the third chapter, we summarize the most important topics of the Schroer model [1]. The next section is divided in three subsections, where in the first subsection we analyze the renormalizability of the massive Schroer model. The second subsection is dedicated to verify the Ward identities and a new set of normalization conditions concerning the derivatives of the massive boson that we called extended Ward Identities. These new conditions are motivated by the requirement that in the adiabatic limit, where the test function $g(x)$ goes to a fixed constant (coupling constant) q, in symbols $g \to q$, we obtain

$$S[g\partial_\mu\phi j^\mu] \to \mathbb{1} \qquad X\Big|_{g\partial_\mu\phi j^\mu} \to X \qquad \psi\Big|_{g\partial_\mu\phi j^\mu} \to \psi_q =: e^{iq\phi} : \psi \tag{1}$$

where S is the S-matrix, X is an observable, i.e. $X \in \{\partial_\mu\phi, j^\mu\}$, and ψ_q is the free dressed Dirac field from [1]. In the last subsection, we prove a theorem stating that the requirements above are indeed satisfied.

[4] Over the last years, infrared divergences were revisited by Strominger and collaborators and organized in this triangle. A extensive review can be found in [32].

1 Quantum Fields

The structure of this chapter is the following: In Section 1, we introduce a collection of mathematical concepts. In Section 2, we discuss the structure of the spacetime. The axiomatic framework of Gårding and Wightman is presented in Sections 3 and 4.

1.1 Mathematical Preamble

This section will cover some topics, mainly about distribution theory, which are going to be used throughout this work and is based in [33–35]. In Physics, distributions first appeared in Quantum Mechanics as Dirac introduced his δ-function and it quickly thrived in the community, founding applications in other areas such as Electrodynamics, Statistical Physics and QFT [36–38].

Definition 1. A Hilbert space $\mathcal{H}$ is a vector space with an inner product that is complete with respect to the norm defined by the inner product.

Definition 2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^{\otimes_{s/a} n}$ the symmetrized (anti-symmetrized) n-fold tensor product $\mathcal{H}^{\otimes_{s/a} n} = \mathcal{H} \otimes_{s/a} \cdots \otimes_{s/a} \mathcal{H}$. Set $\mathcal{H}^0 = \mathbb{C}$ and define

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_{s/a} n} \tag{1.1}$$

where $\mathcal{F}(\mathcal{H})$ is known as the bosonic (fermionic) Fock space over $\mathcal{H}$.

Definition 3. Let $X \subset \mathbb{R}^n$. The support of a continuous function $\varphi : X \to \mathbb{C}$ is the closure of the set of non-vanishing points of φ, that is[1],

$$supp\, \varphi = \overline{\{x \in X | \varphi(x) \neq 0\}} \tag{1.2}$$

Definition 4. Let $X \subset \mathbb{R}^n$ be an open set. The set of the infinitely differentiable functions $\varphi : X \to \mathbb{C}$ with compact support is called space of test functions on X and is denoted by $C_0^\infty(X)$ (or $\mathcal{D}(X)$).

To properly give the definition of a distribution we need to introduce a topology or at least a notion of convergence in $\mathcal{D}$.

Definition 5. A sequence (φ_n) of elements of $\mathcal{D}$ converges to $\varphi \in \mathcal{D}$, in symbols $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$, iff

[1] The bar above the set denotes the closure.

i) The supports of all functions φ_n are contained in the same bounded set, regardless of the n;

ii) The sequence of the derivatives of any order of the functions φ_n converges uniformely to the corresponding derivatives of φ.

Here we only ask for the uniform convergence of each order of differentiation taken separately.

Definition 6. Let $X \subset \mathbb{R}^n$ be an open set. A continuous linear functional $T : C_0^\infty(X) \to \mathbb{C}$ is a *distribution* in X. So, for $\varphi_1, \varphi_2 \in C_0^\infty(X)$, $\lambda \in \mathbb{C}$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(X)$

- $T(\varphi_1 + \lambda\varphi_2) = T(\varphi_1) + \lambda T(\varphi_2)$
- $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi \Rightarrow T(\varphi_n) \underset{n\to\infty}{\longrightarrow} T(\varphi)$

Example 7 (Dirac Delta Distribution)**.** This distribution is defined by

$$\delta(\varphi) = \int dx \delta(x)\varphi(x) = \varphi(0).$$

At a point $a \in \mathbb{R}^n$ it is defined as follows

$$\delta_a(\varphi) = \varphi(a).$$

Definition 8. Let $C^\infty(\mathbb{R}^n)$ be the set of complex functions defined in $\mathbb{R}^n$ that possess all orders of partial derivatives. For $f \in C^\infty(\mathbb{R}^n)$ and the multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we define

$$\partial_\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}\,, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n\,, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Now, we define the Schwartz space[2] as

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \sup_{x\in\mathbb{R}^n} |x^\alpha(\partial_\beta f)(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^n\}. \tag{1.3}$$

Definition 9. A sequence (f_n) of functions of $\mathcal{S}(\mathbb{R}^n)$ converges to $f \in \mathcal{S}(\mathbb{R}^n)$ iff the sequence $(x^\alpha \partial_\beta f_n)$ converges uniformely to $x^\alpha \partial_\beta f$ for every $\alpha, \beta \in \mathbb{N}_0^n$.

Definition 10. A *tempered distribution* is a linear continuous functional over $\mathcal{S}$. As $\mathcal{D} \subset \mathcal{S}$, if $\varphi_n \to \varphi$ in $\mathcal{D}$ then $\varphi_n \to \varphi$ in $\mathcal{S}$, in such manner that a tempered distribution is a linear continuous functional in $\mathcal{D}$ is extendable to a linear continuous functional in $\mathcal{S}$. The space of tempered distributions is denoted by $\mathcal{S}'$.

Theorem 11 (Nuclear Theorem)**.** Let T be a multilinear functional of arguments $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^k)$ which is continuous in each of its arguments while the others are fixed. Then there is a unique distribution $G \in \mathcal{S}'(\mathbb{R}^{k.n})$ in all the variables of $f_1, \dots, f_n$ such that $T(f_1, \dots, f_n) = G(f_1, \dots, f_n)$.

[2] Sometimes it is referred in the literature as the space of the well behaved functions.

1.2 Relativistic spacetime

Historically, Quantum Field Theory was born as an attempt to conciliate Einstein's Special Relativity and Quantum Mechanics in a more satisfactory way than relativistic quantum mechanics. For this reason, we should first give an outline of the structure of spacetime[3]. This section is heavily based on the ideas contained in [40, 41]. An interesting reflection about the nature of physical space is present in [42, 43].

In the theory of relativity, Lorentz transformations are used. The change from Galilean to Lorentz transformations is highly motivated by the incompatibility of Galilean transformations with electrodynamics. A few years after the birth of relativity, Minkowski introduced the concept of spacetime, unifying space and time in an absolute concept regardless of the choice of a referential.

An important notion in spacetime, are the so-called events, which are phenomena occurring in such a small region of spacetime that the dimensions can be neglected. Let $\mathbb{M}$ be the spacetime. After the choosing of a reference frame, an event in $\mathbb{M}$ can be identified by an element of $\mathbb{R}^4$,

$$x = (ct, \vec{x}) = (x^0, x^1, x^2, x^3) \tag{1.4}$$

where c is the speed of light, t is the time and $\vec{x}$ is the position vector relative to the reference frame used to identify $\mathbb{M}$ with $\mathbb{R}^4$. It is possible to define a quadratic quantity[4]

$$\begin{aligned} Q: \quad & \mathbb{M} \times \mathbb{M} \to \mathbb{R} \\ (x, y) \quad & \mapsto Q(x, y) = (x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2. \end{aligned} \tag{1.5}$$

This quantity represents the interval between two events. One can notice that for a luminous signal, $Q(x, y) = 0$. Due to the invariance of the speed of light, this result does not depend on the choice of the reference frame. It is also possible to prove that Q is invariant to changes in the coordinate system.

$$Q(x, y) = Q(x', y') \tag{1.6}$$

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{M} \times \mathbb{M}$ be two pairs of events. We define the following equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1^\mu - y_1^\mu = x_2^\mu - y_2^\mu \qquad \mu = 0, 1, 2, 3. \tag{1.7}$$

This relation divides $\mathbb{M}$ into equivalence classes. The representative of one of this classes is called a four-vector and we denote it $[(x, y)]$. The space of all four-vectors is denoted by $\mathbb{D}(\mathbb{M})$.

[3] A different approach is given in [39] where the construction is based in manifolds and is an excellent alternative literature regarding relativity.

[4] Nondegenerate, symmetric and bilinear form

We can define a general transformation which preserves the spacetime interval Q

$$(Lx)^\mu = a^\mu + \Lambda^\mu_\nu x^\nu \tag{1.8}$$

where $a^\mu \in \mathbb{D}(\mathbb{M})$ and Λ is a Lorentz transformation, i.e., a matrix Λ that satisfies $\eta = \Lambda^T \eta \Lambda$ with $\eta = diag(1, -1, -1, -1)$ being the Minkowski metric. This transformation L is called Poincaré transformation and is a composition of translations, rotations and boosts. Notice that every Poincaré transformation is a linear non-homogeneous transformation.

Since translations and rotations do not alter (1.7), our definition of four-vectors holds in every inertial frame. Furthermore, $Q(x, y)$ only depends on the equivalence class, allowing one to define $Q([x, y]) := Q(x, y)$. It is important to remark that given a reference frame, one can identify $\mathbb{M}$ to the quotient $(\mathbb{M} \times \mathbb{M})/_\sim \doteq \mathbb{D}(\mathbb{M})$, where $\sim$ was defined in (1.7), and both to $\mathbb{R}^4$ equipped with the pseudo-metric η.

We can define the inner product as

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}. \tag{1.9}$$

With this definition, we obtain the notion of orthogonality in spacetime which is very different from the usual Euclidean version [39].

Since this is an extensive subject and for the sake of brevity, as we have already covered the basics of spacetime structure, let us focus on giving a few fundamental definitions.

Definition 12. Let $x \in \mathbb{D}(\mathbb{M})$. Then, x is said to be timelike if $x^2 \equiv x \cdot x > 0$, lightlike if $x^2 = 0$ and spacelike if $x^2 < 0$.

Definition 13. We can now define a region called lightcone. Let $x \in \mathbb{M}$. The future lightcone of x is given by $V_+(x) := \{y \in \mathbb{M} | (y - x)^2 > 0 \wedge (y - x)^0 > 0\}$. In words, this is the set of all time-like future-pointing vectors.

In the same idea, we define the past lightcone $V_-(x) := \{y \in \mathbb{M} | (y - x)^2 > 0 \wedge (y - x)^0 < 0\}$ which is the set of all time-like past-pointing vectors.

The boundary of the future lightcone is $\partial V_+(x) = \{y \in \mathbb{M} | (y-x)^2 = 0 \wedge (y-x)^0 \geq 0\}$. Similarly, $\partial V_-(x) = \{y \in \mathbb{M} | (y - x)^2 = 0 \wedge (y - x)^0 \leq 0\}$ is the boundary of the past lightcone.

Remark. All these regions $(V_\pm, \partial V_\pm)$ are invariant under the proper orthochronous Lorentz group $\mathcal{L}^\uparrow_+$, i.e. $\det \Lambda = 1$ and $\Lambda^0_0 \geq 1$.

Definition 14. A notion of geometric time ordering is that x is later than y, in symbols $x \succeq y$, if $x \notin \overline{V_-(y)}$ and is equivalent to saying that there exists a reference frame such that $x^0 > y^0$.

To close this Section, we present one last Lemma which will be useful below in Section 2.2.

Lemma 15. Let $x, y \in \mathbb{R}^4$ be events in a given inertial frame. Thus, the set

$$P \doteq \{(x,y) : x \succeq y\} \tag{1.10}$$

is open in $\mathbb{R}^4 \times \mathbb{R}^4$.

1.3 Wightman Axioms

The many successes, and failures, of relativistic QFT motivated physicist and mathematicians to investigate its foundations. Moreover, a long list of mathematical problems led Wightman and Gårding to extract general postulates that later became known as the Wightman axioms.

These "axioms" were only named like this to bring attention to their consistency and mathematical rigor. They are well formulated physical requirements constructed to emphasize the spectral condition, relativistic invariance, and locality and will be presented below [44–46].

Structure of the Theory: The space in question is a separable Hilbert space $\mathcal{H}$ and the states are described by unit rays. This space possesses a continuous unitary representation of the non-homogeneous group $SL(2,\mathbb{C})$[5] given by $U(a,A), a \in \mathbb{R}^4, A \in SL(2,\mathbb{C})$ and by its unity we can write $U(a,\mathbb{1}) = e^{iP^\mu a_\mu}$ where P^μ is an unbounded self-adjoint operator, seen as the energy-momentum operator.

Energy-Momentum Spectral Condition: The joint spectrum of P^μ lies in the closed forward cone $\overline{V}_+ \coloneqq \{p_\mu : p^2 \geq 0, p_0 \geq 0\}$, and $P^\mu P_\mu$ is the mass operator[6].

There is a vector Ω which is translation invariant in $\mathcal{H}$, $U(a,\mathbb{1})\Omega = \Omega$, unique up to a constant. This vector is named vacuum state.

Field Operators: Fields are operator-valued distributions in $\mathcal{H}$ [47]. For each $f \in \mathcal{S}(\mathbb{R}^4)$, there exists a set of operators along with its adjoints defined in a domain $\mathcal{D}$ dense in $\mathcal{H}$[7]. Let $\{\varphi_1, \ldots, \varphi_n\}$ be fields. The domain $\mathcal{D}$ always contains a domain $\mathcal{D}_0 \coloneqq \text{span}\{\varphi_{i_1}(f_1)\ldots\varphi_{i_n}(f_n)\Omega; n > 0\}$, i.e. obtained by the application of polynomials of smeared fields on the vacuum state (cyclicity of the vacuum [48]). Further details about the domain and operator in [44].

5 Special Linear group with complex entries $SL(2,\mathbb{C}) \doteq \{A, n \times n \text{ matrix}, A_{ij} \in \mathbb{C}, det(A) = 1\}$. This is known to be the universal covering of $SO_+^\uparrow(1,3)$ which is the proper orthochronous Lorentz group.

6 This spectral condition is the relativistic version of the condition that the Hamiltonian is bounded from below.

7 A set S is said to be dense in $\mathcal{H}$ if, for each vector $\varphi \in \mathcal{H}$ and $\epsilon > 0$, there exists a vector $\psi \in S$ such that $|\varphi - \psi| < \epsilon$.

Covariance: The fields transform under the Poincaré transformations $U(a, A)$

$$U(a, A)\varphi_i(x)U(a, A)^{-1} = S(A^{-1})_{ij}\varphi_j(\Lambda x + a) \tag{1.11}$$

with $S(A)$ being a finite dimensional representation of $SL(2, \mathbb{C})$.

Given the fact that all finite-dimensional representations of $SL(2, \mathbb{C})$ are the sum of irreducible representations, one can take into account the components of an irreducible representations as part of the field and distinguish different fields within the theory by looking at different irreducible representations. The same idea is used when grouping together components transformed by a reducible representation in a field.

Causality: Let $f, g \in \mathcal{S}(\mathbb{R}^4)$. If the support of f, g are spacelike separated, $(x - y)^2 < 0$, then

$$[\varphi_i(f), \varphi_j(g)]_\pm = 0 \qquad [\varphi_i(f), \varphi_j(g)^*]_\pm = 0 \tag{1.12}$$

$\forall i, j$ when applied to a vector in $\mathcal{D}$. The plus or minus sign means that the fields either anticommute or commute. They anticommute iff both fields are fermions.

1.4 Wightman Functions

In this section we will discuss the properties of the vacuum expectation values which are also known as Wightman functions[8]. We note that the axiom regarding field operators implies that

$$\langle \varphi_{i_1}(f_1) \cdots \varphi_{i_n}(f_n) \rangle \doteq (\Omega, \varphi_{i_1}(f_1) \cdots \varphi_{i_n}(f_n)\Omega) \tag{1.13}$$

exists and is a separately continuous multilinear functional of the arguments $f_1, \ldots f_n$ as they vary over $\mathcal{S}(\mathbb{R}^n)$. From Theorem 11, we have that this functional can be uniquely extended to be a tempered distribution of the n four-vectors $x_1, \ldots, x_n$. We shall denote $w_{i_1,\ldots,i_n}(x_1, \ldots, x_n) \equiv \langle \varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n) \rangle$.

(W1) Covariance: For $(a, A) \in \mathcal{P}_+^\uparrow$ there holds

$$w_{i_1 \ldots i_n}(x_1, \ldots, x_n) = S(A^{-1})_{i_1 j_1} \ldots S(A^{-1})_{i_n j_n} w_{j_1 \ldots j_n}(a + Ax_1, \ldots, a + Ax_n) \tag{1.14}$$

with S a finite dimensional representation of $SL(2, \mathbb{C})$ (or of the Lorentz group if the spin is integer, i.e. a boson).

We need to introduce a convention before the next property. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^{2n})$ is given by

$$\tilde{f}(p) \doteq \int d^n x f(x) e^{ip \cdot x} \tag{1.15}$$

[8] These are detailed in [44][Chapter 3.3]

with $p \cdot x = \sum_{k=1}^{n} (p_k)_\mu (x_k)^\mu$.

We will describe the next properties for one scalar field (boson).

(W2) Spectral property: The support of the Fourier transform $\tilde{w}$ of w is contained in the product of forward lightcones

$$\tilde{w}(p_1, \ldots, p_n) = 0, \quad \text{if, for some } j,\ p_j \notin \overline{V}_+. \tag{1.16}$$

(W3) Locality: From causality one has

$$w(x_1, \ldots, x_j, \ldots, x_l, \ldots, x_n) = w(x_1, \ldots, x_l, \ldots, x_j, \ldots, x_n), \text{ if } (x_j - x_l)^2 < 0 \tag{1.17}$$

(W4) Positivity: Consider the Hilbert space structure required in the first axiom. For any terminating sequence $f = (f_0, \ldots, f_n),\ f_i \in \mathcal{S}(\mathbb{R}^{2i})$, one has

$$\sum_{j,k} \int dx_1 \cdots dx_j dy_1 \cdots dy_k \overline{f}_j(x_j, \ldots, x_1) f_k(y_1, \ldots, y_k) w(x_1, \ldots, x_j, y_1, \ldots, y_k) \geq 0 \tag{1.18}$$

This is equivalent to the positivity of the norm of any state of the form

$$\Psi_f = f_0 \Omega + \varphi(f_1)\Omega + \varphi(f_2^{(1)})\varphi(f_2^{(2)})\Omega + \ldots \tag{1.19}$$

where $f = (f_0, \ldots, f_N),\ f_j = \bigotimes_{k=1}^{j} f_j^{(k)},\ f_j^{(k)} \in \mathcal{S}(\mathbb{R}^2)$.

(W5) Cluster Property: For any spacelike vector a and for $\lambda \to \infty$, the uniqueness of the translationally invariant state is equivalent to

$$w(x_1, \ldots, x_j, x_{j+1} + \lambda a, \ldots, x_n + \lambda a) \to w(x_1, \ldots, x_j) w(x_{j+1}, \ldots, x_n) \tag{1.20}$$

here the convergence has to be understood in the sense of distributions.

This last property says that the correlation function of two monomials of smeared fields, known as clusters, factorize in the limit of infinite spacelike distance between the two clusters. This property plays a crucial role for the existence of asymptotic free fields and the construction of the S-matrix.

The cluster property implies that if two clusters B_1, B_2 are localized in bounded regions $\mathcal{O}_1, \mathcal{O}_2$, respectively, the state vectors $B_1\,\Omega, B_2\,\Omega$ become orthogonal, apart from their vacuum component, in the limit in which the spacelike separation between their localization regions becomes infinite.

Theorem 16 (Wightman Reconstruction Theorem)**.** Let $(w_n)_{n\in\mathbb{N}}$ be a family of distributions $w_n \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfying **(W1 - W5)**. Then, there exists a separable Hilbert space $\mathcal{H}$, a continuous unitary representation U of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$ with a unique invariant vector Ω, and a hermitian scalar Wightman field φ which is covariant under U, such that

$$\langle \Omega, \varphi(f_1)\dots\varphi(f_n)\Omega\rangle = w_n(f_1 \otimes \cdots \otimes f_n). \tag{1.21}$$

Moreover, any other field theory with these vacuum expectation values is unitary equivalent to this one.

Definition 17 (Wick product)**.** We define the Wick products of free fields (normal ordering) by writing the free field as creation and annihilation operators, $\varphi(x) = a(x) + a^*(x)$, and bringing the annihilation operators to the left hand side of the creation operators,

$$:\varphi(x_1)\dots\varphi(x_n):=\sum_{I\subset\{1,\dots,n\}}\prod_{i\in I} a^*(x_i)\prod_{j\notin I} a(x_j)$$

Theorem 18 (Wick's Theorem)**.** Let $\varphi_1,\dots,\varphi_n$ be free fields. Then, we have

a) The weak form of the theorem:

$$\frac{:\varphi_1^{k_1}:\cdots:\varphi_n^{k_n}:}{k_1!\cdots k_n!} = \sum_{l_1,\dots,l_n} \frac{:\varphi_1^{l_1}\cdots\phi_n^{l_n}:}{l_1!\cdots l_n!}\frac{\langle:\varphi_1^{k_1-l_1}:\cdots:\varphi_n^{k_n-l_n}:\rangle}{(k_1-l_1)!\cdots(k_n-l_n)!} \tag{1.22}$$

with $0 \le l_i \le k_i$.

b) The strong form:

$$\langle:\varphi_1^{k_1}:\cdots:\varphi_n^{k_n}:\rangle = \sum_{\substack{l_{ij}(i<j)\\ \sum_{i\neq j} l_{ij}=k_i}} \prod_{i<j}\frac{\langle\varphi_i\varphi_j\rangle^{l_{ij}}}{l_{ij}!} \tag{1.23}$$

Define W_i as a polynomial of a free field $\varphi_i^{k_i}$. Then, we can rewrite a) as

$$W_1\cdots W_n = \sum :W_1'\cdots W_n': \langle W_1''\cdots W_n''\rangle$$

where the sum goes over the possible factorizations of each W_i.

2 The Epstein-Glaser Scheme

The most successful approach to interacting field theories is the definition of the S-matrix by a formal power series in the test function g. Unfortunately, this approach has a longstanding problem, the so-called Infrared(IR) and Ultraviolet(UV) divergences. The UV divergences can be treated by various methods of renormalization. Up to now, IR divergences can not be completely treated.

A mathematically rigorous method of perturbative construction is the causal perturbation theory, which was elaborated by Epstein and Glaser [49] based on the ideas of Stückelberg, Bogoliubov and Shirkov [50]. In this approach, the Bogoliubov S-matrix and the interacting fields are constructed using time-ordered products of (Wick polynomials of) free fields. One first specifies the set of axioms that are satisfied by the time-ordered products and then performs an inductive construction. The physical S-matrix is obtained by taking the so-called adiabatic limit of the Bogoliubov S-matrix. We discuss the construction of both the S-matrix and the interacting observable fields in Section 3.3.

The solution of the UV problem in this framework consists of finding an extension of distributions which are initially defined only on a suitable subspace of the space of all test functions. The freedom in renormalization is the consequence of the non-uniqueness of the extension. This chapter serves as an intuitive introductory contact to the results in the next chapter and follows [51–53]. The infrared problem remains in the adiabatic limit [54, 55], but it will not appear here since the massless limit of the boson ϕ is not taken.

2.1 Time-Ordered Products

The time-ordered products $T[W_1(x_1)\cdots W_n(x_n)]$, with $W_k(x)$ being Wick polynomials of free fields, are the building blocks to construct interacting fields and obtain the S-matrix. They are multilinear maps with C^∞ functions as coefficients, symmetrical operator-valued distributions on the dense domain $\mathcal{D}$ [1] which satisfies the following axioms

(P1) Translation Covariance :

$$U(a,\mathbb{1})T[W_1(x_1)\dots W_n(x_n)]U(a,\mathbb{1})^{-1} = T[W_1(a+x_1)\dots W_n(a+x_n)] \tag{2.1}$$

where $(a,\mathbb{1})\in\mathcal{P}_+^\uparrow$ and U is a unitary positive energy representation of the Poincaré group.

[1] This is usually referred as the well-posedness property **(P0)** of T-products.

(P2) Causality : If $x_i \succeq x_j \ \forall i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$, then

$$T[W_1(x_1) \cdots W_n(x_n)] = T[W_1(x_1) \cdots W_k(x_k)] T[W_{k+1}(x_{k+1}) \cdots W_n(x_n)] \quad (2.2)$$

(P3) Graded Symmetry :

$$T[W_1(x_1) \cdots W_n(x_n)] = (-1)^{\mathrm{f}(\pi)} T[W_{\pi(1)}(x_{\pi(1)}) \cdots W_{\pi(n)}(x_{\pi(n)})] \quad (2.3)$$

where $\pi \in \mathrm{S}_n$ is any permutation of the set $\{1, \ldots, n\}$ and $\mathrm{f}(\pi)$ is the number of transpositions in π involving two fermion fields $\{\psi, \overline{\psi}, :\psi\phi:, :\overline{\psi}\phi:, :\psi\partial\!\!\!/\phi:, :\partial\!\!\!/\phi\overline{\psi}:\}$.

We also require that

$$T[\emptyset] \doteq \mathbb{1} \qquad T[W(x)] \doteq :W(x): \ . \quad (2.4)$$

If the time-ordered products of less than n factors are everywhere defined, the time ordered product is uniquely determined up to the total diagonal $D_n \doteq \{(x_1, \ldots, x_n), x_i \in \mathbb{R}^2 | x_1 = \cdots = x_n\}$. Therefore, the renormalization problem, in this scheme, amounts as the problem of extending the $n-$th order time-ordered products to the diagonal D_n.

2.2 Inductive Construction

In this section we formulate the inductive construction of the time-ordered products[2]. This procedure was first presented by Epstein and Glaser [49]. First, we assume that all time-ordered products have been constructed up to $n-1$ arguments, as defined in the last section. We begin with the inductive construction up to D_n. For each proper subset $I \subset \{1, \ldots, n\}$, let

$$U_I := \{x \in \mathbb{R}^{2n} \setminus D_n | x_i \succeq x_j \text{ for all } (i,j) \in I \times I^c\}. \quad (2.5)$$

Proposition 19. Let D_n be the total diagonal of $\mathbb{R}^{2n}$ and a proper subset $I \subset \{1, \ldots n\}$. Then, U_I is an open set in $\mathbb{R}^{2n}$.

Proof. Let $I \subset \{1, \ldots, n\}$ be a proper subset. Given $x \in \mathbb{R}^{2n} \setminus D_n$, we write $x = (x_1, \ldots, x_n)$, where we are considering the decomposition $\mathbb{R}^{2n} = (\mathbb{R}^2)^n$. Consider now the set

$$P_j^k := \{(x_1, \ldots, x_n) | x_j \succeq x_k\}, \quad k \in \{1, \ldots, n-1\} \setminus \{j\} \quad (2.6)$$

By Lemma 15, each set $P_j^k, \ldots, P_j^{n-1}$ is open. Thus, $X_1 = \cap_{i=k}^n P_1^i$ is open. This means that $X_j = \cap_{i=k}^n P_j^i$ is also open. Therefore,

$$U_I = X_1 \cap \cdots \cap X_k \quad (2.7)$$

is also open since it is a finite intersection of open sets. □

2 For the string-localized version see [56].

The family of open sets U_I, when I runs through the proper subsets of $\{1, \dots, n\}$, is a covering of $\mathbb{R}^{2n} \setminus D_n$, that is, every $x \in \mathbb{R}^{2n} \setminus D_n$ is in some U_I, see [51, Lemma 4.1]. On U_I, the factorization property implies that

$$T[W_1(x_1) \cdots W_n(x_n)] \doteq T_n(x_1, \dots, x_n) = T_{|I|}(I) T_{n-|I|}(I^c) \tag{2.8}$$

which is known by hypothesis since $1 \leq |I| \leq n-1$, $|I|$ is the number of elements of I, and I^c is the complement of I in $\{1, \dots, n\}$.

Lemma 20 (Uniqueness outside D_n)**.** Let $\mathcal{W}$ be a linear space of Wick polynomials with lower or equal order with respect to the interacting Lagrangian of the model in question. If the time-ordered products are known for up to $n-1$ Wick polynomials in $\mathcal{W}$, then the time-ordered product is fixed and well defined on the complement of D_n by the above formula (2.8).

Proof. First we need to prove that U_I is indeed a covering. For this we refer the reader to [51, Lemma 4.1]. It remains only to show that T_n does not depend on the choice of I. To this end, let J be another set such that $x_i \precsim x_j$ for all $i \in J$, $j \in J^c$. Then write[3] $I = I \cap J \,\dot{\cup}\, I \cap J^c$, hence $T(I) = T(I \cap J) T(I \cap J^c)$ by equation (2.2). Similarly, $I^c = I^c \cap J \,\dot{\cup}\, I^c \cap J^c$ and hence $T(I^c) = T(I^c \cap J) T(I^c \cap J^c)$. Thus,

$$T(\{1, \dots, n\}) = T(I \cap J) T(I \cap J^c) T(I^c \cap J) T(I^c \cap J^c).$$

Let now $i \in I \cap J^c$ and $j \in I^c \cap J$. Then, x_i is larger and smaller than x_j. But this implies that x_i and x_j are spacelike separated. Therefore, the two factors in the middle commute. Applying the factorization property (2.2) again, yields $T(J)T(J^c)$. This establishes independence of the choice of I in (2.8). $\square$

This inductive construction of T-products leaves us with operator-valued distributions well defined in $\mathcal{D}(\mathbb{R}^{2n} \setminus D_n)$ satisfying **(P1-P3)**. It remains to extend these products to the total diagonal D_n, and this will be done below at Section 3.2. The Epstein-Glaser scheme requires that the pointwise products of Wick polynomials $W_i(x_i)$ with translational invariant numerical distributions t to be well defined. This product indeed exists and is the result of a Theorem, known as *Theorem 0* in [49]. There is also a microlocal version due to Brunetti and Fredenhagen [51, Theorem 3.1]. Hence, the construction of T_n reduces to finding an extension of the numerical distribution t across the total diagonal D_n. From translational invariance, the numerical distributions depend only on the relative coordinates and we can write the total diagonal as the origin of these coordinates.

[3] The symbol $\dot{\cup}$ stands for the disjoint union of two sets.

2.3 Extension of Distributions

The last section ended with a problem, how to extend the numerical distribution t across the origin. This problem requires the introduction of a quantity that measures the singularity of distributions at the origin. First, we define the rescaling of a distribution and then introduce the definition of a well-known quantity, the scaling degree.

Definition 21 (Distribution Rescaling)**.** Let t be a distribution on $\mathbb{R}^k$. The rescaled distribution t_λ, $\lambda > 0$ is defined as

$$t_\lambda(f) := t(f^\lambda) \text{ with } f^\lambda(x) := \lambda^{-k} f(\lambda^{-1}x). \tag{2.9}$$

Definition 22 (Steinmann scaling degree)**.** Let $t \in \mathcal{D}'(\mathbb{R}^k)$ and $\lambda > 0$. The *scaling degree* of t with respect to the origin is[4]

$$sd(t) := \inf\{s \in \mathbb{R} : \lim_{\lambda \to 0} \lambda^s t_\lambda(f) = 0\}. \tag{2.10}$$

By definition, $\inf \emptyset := \infty$, i.e. if there is no such s, the scaling degree is said to be infinite. At this point, some examples may be helpful.

Example 23. Let $\delta \in \mathcal{D}'(\mathbb{R}^k)$. Then $\delta(\lambda x) = |\lambda|^{-k}\delta(x)$, so $sd(\delta) = k$.

Example 24. Let us now calculate the scaling degree of the two-point function in $d = 1+1$. The invariant measure rescales as

$$d\mu_m(\lambda^{-1}p) = \frac{d^2\lambda^{-1}\vec{p}}{2\sqrt{\lambda^{-2}|\vec{p}|^2 + m^2}} = \lambda^0 \frac{d^2\vec{p}}{2\sqrt{|\vec{p}|^2 + (\lambda m)^2}} = d\mu_{\lambda m}(p) \tag{2.11}$$

where $p = (p^0, \vec{p})$. Then, the rescaled two-point function is,

$$w_{m,\lambda}(x) = w_m(\lambda x) = (2\pi)^{-1} \int_{H_m^+} d\mu_m(p) e^{-i\lambda p.x} = \lambda^0 \int_{H_m^+} d\mu_{\lambda m}(p) e^{-ip.x} = \lambda^0 w_{\lambda m}(x) \tag{2.12}$$

where H_m^+ is the mass shell. Since $\lim_{\lambda \to 0} w_{\lambda m} = w_0$ is logarithmic divergent the scaling degree of w is 0.

The scaling degree have some interesting properties which we state in the following proposition [51, Lemma 5.1].

Proposition 25. Let $X \subset \mathbb{R}^k$ and $u, w \in \mathcal{D}'(X)$ distributions, and α a multi-index,

(i) $sd(\partial^\alpha u) = sd(u) + |\alpha|$.

(ii) $sd(x^\alpha u) = sd(u) - |\alpha|$

(iii) $sd(u \otimes w) = sd(u) + sd(w)$, where $\otimes$ denotes the tensor product of distributions.

(iv) $sd(f) \leq 0$, $\quad sd(fw) \leq sd(w)$, $f \in \mathcal{D}(\mathbb{R}^k)$.

[4] This defining condition is to be understood in the sense of distributions.

Theorem 26. Let $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$. Then,

(i) If $sd(t^0) < k$, there is a unique extension $t \in \mathcal{D}'(\mathbb{R}^k)$ fulfilling the condition $sd(t^0) = sd(t)$.

(ii) If $k \leq sd(t^0) < \infty$, there are several extensions $t \in \mathcal{D}'(\mathbb{R}^k)$ satisfying the condition $sd(t^0) = sd(t)$. In this case, given a particular solution $\bar{t}$, the general solution is of the form,

$$t = \bar{t} + \sum_{|\alpha| \leq sd(t^0)-n} c_\alpha \partial^\alpha \delta_{(n)} \tag{2.13}$$

with the multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}_0^l$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l$.

In (2.14), the addition of a term $\sum_{|\alpha| \leq sd(t^0)-k} c_\alpha \partial^\alpha \delta_{(n)}$ is called a finite renormalization.

2.4 Normalization Conditions

From Theorem 26, there is some freedom of choice of the extension to the origin, and we formulate conditions to restrict them in this section. The first normalization condition implements the conservation of Poincaré covariance. Let U be a unitary positive energy representation of the Poincaré group $\mathcal{P}_+^\uparrow$(or its universal covering for fermions). Then, we require

$$\textbf{(N1)} \quad \begin{aligned} &U(a,\Lambda)T[W_1(x_1)\cdots W_n(x_n)]U(a,\Lambda)^{-1} = \\ &\qquad T[U(a,\Lambda)W_1(x_1)U(a,\Lambda)^{-1}\cdots U(a,\Lambda)W_n(x_n)U(a,\Lambda)^{-1}] \end{aligned} \tag{2.14}$$

where $(a,\Lambda) \in \mathcal{P}_+^\uparrow$. We can see that **(N1)** is an extension of property **(P1)**. It is remarked by Epstein-Glaser in [49] that property **(P1)**, i.e. translational covariance, is crucial for the causal construction, and is used in their Theorem 0. Brunetti and Fredenhagen [51] have shown that this condition can be replaced by a weaker one, spectrality. This condition is connected to wave front set properties required in their work. The next condition comes from unitarity,

$$\textbf{(N2)} \qquad T[W_1(x_1)\cdots W_n(x_n)]^* = \sum_{P \in \mathrm{Part}\{1,\ldots,n\}} (-1)^{|P|+n} \prod_{p \in P} T[W_i(x_i)^*, i \in p] \tag{2.15}$$

being * the adjoint on $\mathcal{D}$.

In order to reduce the arbitrariness of time-ordered products we require the causal Wick expansion and we will set some notation. We can factorize a Wick polynomial $W \in \{\psi, \overline{\psi}, \phi\psi, \phi\overline{\psi}, j^\mu, \phi j^\mu\}$ as

$$W =: W'W'' :$$

where W', W'' are sub polynomials of W. Note that this is not a unique factorization[5]. Then, we can write the Wick expansion as

$$\textbf{(N3)} \qquad T[W_1 \cdots W_n] = \sum_G \underbrace{: W_1' \cdots W_n' :}_{W_G} \underbrace{\langle T W_1'' \cdots W_n'' \rangle}_{t_G} \tag{2.16}$$

where the sum goes over the possible factorizations of each W_k. Each term can be associated to a graph G.

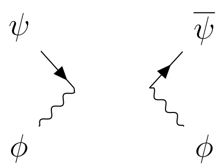
This association might seem abstract, so it is interesting to look at a simple example. We will see more details on the graphs appearing in the Wick expansion in Section 3.2.1.

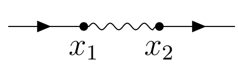
Notation: From now on we denote $W_k := W_k(x_k)$. We also omit the dependence of the variable x, i.e. $\psi := \psi(x)$, $\phi_1 := \phi_1(x_1)$, $\partial_\mu := \frac{\partial}{\partial x^\mu}$.

Example. Consider the Wick expansion of $T[\phi_1\psi_1\overline{\psi}_2\phi_2]$.

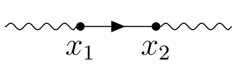
$$\begin{aligned} T[:\phi_1\psi_1 :: \overline{\psi}_2\phi_2 :] &=: \phi_1\psi_1\overline{\psi}_2\phi_2 : + : \psi_1\overline{\psi}_2 : \langle T : \phi_1\phi_2 :\rangle + : \phi_1\phi_2 : \langle T : \psi_1\overline{\psi}_2 :\rangle + \langle T\phi_1\psi_1\overline{\psi}_2\phi_2\rangle \\ &= \sum_G : W_1'W_2' : \underbrace{\langle T W_1'' W_2'' \rangle}_{t_G} \end{aligned}$$

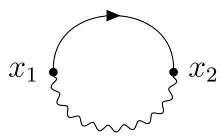
We can represent these terms as

$=$ 

$: \phi_1\psi_1\overline{\psi}_2\phi_2 : +$ 

$: \psi_1\overline{\psi}_2 : \langle T : \phi_1\phi_2 :\rangle +$



$: \phi_1\phi_2 : \langle T : \psi_1\overline{\psi}_2 :\rangle +$ 

$\langle T\phi_1\psi_1\overline{\psi}_2\phi_2\rangle$

We also require condition **(N4)** from [57]. We will write it out for the massive Schroer model. Let ψ be the free Dirac field and ϕ the massive free scalar field. We consider $W_i \in \mathcal{W}$ sub polynomials as defined in (3.6). Then the normalization condition reads

$$\textbf{(N4)} \quad \begin{aligned} (\Box + m^2)T[\phi W_1 \cdots W_n] &= -i\sum_{k=1}^n \delta(x - x_k) T[W_1 \cdots \frac{\partial W_k}{\partial \phi} \cdots W_n] \\ (\partial\!\!\!/ + iM)T[\psi W_1 \cdots W_n] &= \sum_{k=1}^n (-1)^{\mathbf{f}(W_1 \cdots W_{k-1})} \delta(x - x_k) T[W_1 \cdots \frac{\partial W_k}{\partial \overline{\psi}} \cdots W_n] \\ T[W_1 \cdots W_n\overline{\psi}](\overleftarrow{\partial\!\!\!/} - iM) &= -\sum_{k=1}^n (-1)^{\mathbf{f}(W_{k+1} \cdots W_n)} \delta(x_k - x) T[W_1 \cdots \frac{\partial W_k}{\partial \psi} \cdots W_n] \end{aligned} \tag{2.17}$$

[5] For example, take j^μ. These factorizations are possible $W' = j^\mu, \overline{\psi}, \psi$ and 1.

where

$$f(W_1 \cdots W_{k-1}) \doteq \begin{cases} 0, & k=1 \\ f(W_1), & k=2 \\ \sum_{i=1}^{k} \mathrm{f}(W_i), & k>2 \end{cases}$$

with $\mathrm{f}(W_i) = \begin{cases} 1, & W_i \text{ fermion} \\ 0, & W_i \text{ boson} \end{cases}$.

This condition uniquely determines time-ordered products with additional free field factors [57], namely

$$\begin{aligned} \langle T[W_1 \cdots W_n \phi]\rangle &= i \sum_{k=1}^{n} \Delta^F(x - x_k) \langle T[W_1 \cdots \frac{\partial W_k}{\partial \phi} \cdots W_n]\rangle \\ \textbf{(N4')} \quad \langle T[\psi W_1 \cdots W_n]\rangle &= i \sum_{k=1}^{n} (-1)^{\mathbf{f}(W_1 \cdots W_{k-1})} S^F(x - x_k) \langle T[W_1 \cdots \frac{\partial W_k}{\partial \overline{\psi}} \cdots W_n]\rangle \\ \langle T[W_1 \cdots W_n \overline{\psi}]\rangle &= i \sum_{k=1}^{n} (-1)^{\mathbf{f}(W_{k+1} \cdots W_n)} S^F(x_k - x) \langle T[W_1 \cdots \frac{\partial W_k}{\partial \psi} \cdots W_n]\rangle \end{aligned} \tag{2.18}$$

where $iS^F_{\alpha\beta} = \langle T\psi_\alpha \overline{\psi}'_\beta\rangle$, and $i\Delta^F = \langle T\phi\phi'\rangle$, where ϕ is the free scalar field with mass m.

3 Schroer Model

In this chapter, we make a brief revision of the Schroer model in the first section. Afterwards, we analyze the renormalizability of the massive model and verify the Ward Identities. Then, we show a new normalization condition that we called extended Ward Identities in Section 3.2.3. In the last section, we construct perturbatively the S-matrix, the interacting observable fields, and show that the construction coincides with the exact solution.

3.1 The Formulation of the massive Schroer Model

In the '60s, two-dimensional toy models of QED appeared and became really popular, as they proved to be a useful test laboratory[1]. In QFT, there is a class of models that present a coupling between fermionic currents and derivatives of scalar or pseudo-scalar fields in the classical Lagrangian. These models are called derivative coupling models. The Schroer, Thirring, Schwinger and Rothe-Stamatescu models provide well-known examples of this class of models [1, 59–62]. For an alternative presentation of the Schroer model and a refreshed analysis, see [31].

Let ϕ be a massive scalar field and ψ the Dirac field. Then, the massive Schroer model is defined by the following Lagrangian

$$\mathcal{L}_{Schroer} = \mathcal{L}^0_\phi + \mathcal{L}^0_\psi + \mathcal{L}_{int} \tag{3.1}$$

where

$$\mathcal{L}^0_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \qquad \mathcal{L}^0_\psi = \overline{\psi}(i\partial\!\!\!/ - M)\psi \qquad \mathcal{L}_{int} = q\partial_\mu\phi\overline{\psi}\gamma^\mu\psi.$$

The equations of motion are given by

$$(\Box + m^2)\phi = 0 \qquad (i\partial\!\!\!/ - M)\psi_q(x) = -q\partial_\mu\phi(x)\gamma^\mu\psi_q(x) \tag{3.2}$$

The classic solution has the form

$$\psi_q(x) = e^{iq\phi(x)}\psi(x) \tag{3.3}$$

with ψ a free Dirac field of mass M, ϕ a free scalar field of mass m, and $q \in \mathbb{R}$ is the coupling constant. As one may expect from the discussion of the previous sections, the UV singularities of quantum fields require much more attention for the definition of the

[1] The construction and discussion of some two-dimensional models can be found in [58].

equations and solutions. The general Wightman framework discussed above provides the clear guide for the solution of the mathematical problems which arise. We may then take

$$\psi_q(x) =: e^{iq\phi(x)} : \psi(x) \tag{3.4}$$

as an operator solution instead of the solution (3.3) that only makes sense for classic fields. This is the exact solution of the model. The obtained solution is a well defined operator-valued non-tempered distribution, with well defined Wightman functions. The Schroer model from [1] is then obtained when we set the mass of the boson ϕ to zero, i.e. the limit $m \to 0$.

In the case $m = 0$, we have that a few immediate results appear due to this solution and we will mention two of them. The first is that the Hilbert space $\mathcal{H}$ obtained from the reconstruction is different from the Wigner-Fock space of the original particles. The central consequence is the fact that there are no one-particle states of mass M, in the sense of Wigner, in $\mathcal{H}$. These states are called infraparticles.

3.2 Perturbation Theory

In this section, we are interested in obtaining the massive Schroer model perturbatively within the Epstein-Glaser scheme from an interaction between the free Dirac field and the massive boson $\partial_\mu\phi$. The massless case presents infrared problems and we will not treat this in our work. The interaction density in the model is

$$\mathcal{L}(x) \doteq (\partial_\mu\phi j^\mu) = \partial_\mu(\phi j^\mu) \tag{3.5}$$

where j^μ is the Dirac current. Notice that here we begin by setting the interaction of the quantized fields differently from the Schroer model.

3.2.1 Renormalizability of the Model

The proof of renormalizability in this work was divided into two steps, the inductive construction and the extension across D_n. The first part was done in Section 2.2. Now, it remains to extend our T_n from any of the open sets U_I across the total diagonal D_n. If relative coordinates are employed, then the diagonal coincides with the origin of $\mathbb{R}^{2(n-1)}$. As seen in Section 2.2, it suffices to extend the numerical distributions t_G appearing in its Wick expansion (2.16). The task is thus to determine their scaling degrees. To this end, consider the set of Wick polynomials of $V^\mu \doteq\, : \phi j^\mu :$ and of the form $\phi^r\overline{\psi}, \phi^r\psi$, $r \in \mathbb{N}$,

$$\mathcal{W} \doteq \{\phi, \psi, \overline{\psi}, j^\mu, \phi\psi, \phi\overline{\psi}, \phi j^\mu, \phi^2\psi, \phi^2\overline{\psi}, \phi^3\psi, \dots\}. \tag{3.6}$$

The study of these polynomials will help us understand the behavior of the time-ordered products without derivatives that appear in the massive Schroer model. This is a preparatory step and the importance of this analysis will become evident later on, as we enunciate Theorem 33. With the help of this analysis we also show that our model is completely defined modulo one vacuum expectation value.

Before we move on, let us elaborate on the graphs appearing in the Wick expansion. The points in the Minkowski space $\mathbb{M}$, i.e. events, are represented by the vertices and fields are represented by lines. Each vertex of the graph can have at most one factor ψ, one $\overline{\psi}$, and k boson lines, which is shown by the first graph in Figure 1.

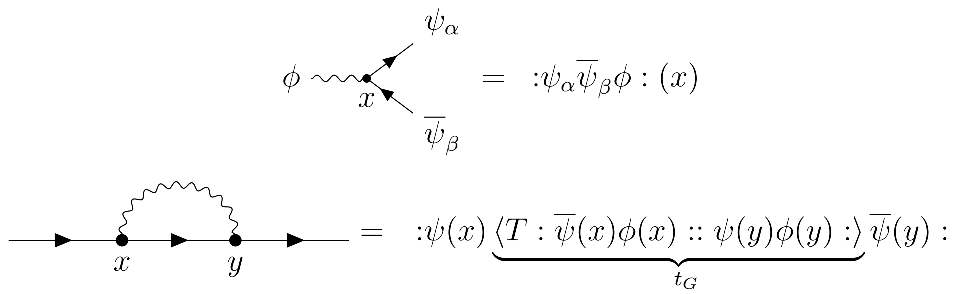


Figure 1 – Graphical representations of time-ordered products.

These lines coming out of vertices can connect to external lines of other vertices and form internal lines of the graph, an example is the second graph of Figure 1. The external lines of the graph represent real particles and the internal lines represent virtual particles.

Heuristically, the bound on $sd(t_G)$ can be found by Wick expanding the ordinary product $W_1 \cdots W_n$ instead of the T product. In this case, the strong form of Wick's Theorem can be used,

$$w = \langle W_1 \cdots W_n \rangle = \sum_{a_{ij}, b_{ij}} \prod_{i<j} w_{\psi\overline{\psi}}(x_i, x_j)^{a_{ij}} w_{\phi\phi}(x_i, x_j)^{b_{ij}} \tag{3.7}$$

where $a_{ij} \in \{0, 1\}$ and $b_{ij} \in \mathbb{N}$. The powers of ϕ increases the number of $w_{\phi\phi}$ and consequently the number of internal boson lines. To obtain the desired results, we need to calculate the scaling degree of the two-point function. This was done in example 24.

Considering that $sd(w_{\phi\phi}) = 0$ and $sd(w_{\psi\overline{\psi}}) = 1$[2] are the scaling degrees w.r.t. the origin, then, the scaling degree of the numerical distribution w in (3.7) is given by

$$sd(w) = i_{\psi\overline{\psi}} \tag{3.8}$$

with $i_{\psi\overline{\psi}}$ being the number of internal $\psi\overline{\psi}$-lines. Since we have n monomials and ψ is always linear we have that ψ appears at most n times in the product $W_1 \cdots W_n$. Using this fact, we arrive at the following relation

$$f + i_{\psi\overline{\psi}} \leq n \tag{3.9}$$

[2] Using the fact that $\langle \psi\overline{\psi} \rangle = (i\partial\!\!\!/ - M)\langle \varphi\varphi \rangle$ where φ is a massive scalar free field, and $sd(\partial^\alpha u) = sd(u) + |\alpha|$.

where f is the number of external ψ-lines in the graph associated with the VEV. Applying this relation we obtain

$$sd(w) \leq n - f. \tag{3.10}$$

With this bound in mind, we can try to obtain the scaling degree of t_G. To this end, we will proceed with an inductive construction à la Epstein-Glaser. The Wick expansion (2.16) of time-ordered product $T[W_1(x_1)\cdots W_n(x_n)]$ with each $W_i \in \mathcal{W}$ can be written as follows. Let $\mathcal{G}^n_{f,\bar{f},b}$ be the set of graphs with n vertices, f external ψ-lines, $\bar{f}$ external $\overline{\psi}$-lines and b external ϕ-lines. Then, the Wick expansion (2.16) reads

$$T[W_1\cdots W_n] = \sum_{f,\bar{f},b}\sum_{G\in\mathcal{G}^n_{f,\bar{f},b}} t_G(\underline{x})W_G(\underline{x}) \tag{3.11}$$

where t_G is a numerical distribution and W_G is the Wick ordered product of f factors ψ, $\bar{f}$ factors $\overline{\psi}$, and b factors ϕ.

Proposition 27. Let G be a graph appearing in the Wick expansion of the time-ordered product T_n given in Eq.(3.11) with $W_i \in \mathcal{W}$ and t_G the numerical distribution associated with G. The scaling degree of t_G with respect to D_n is, for $n \geq 1$,

$$sd(t_G) \leq n - f \tag{3.12}$$

Proof. First, recall that T_n factorizes as (2.8), that is $T_n(X) = T(I)T(I^c)$ on U_I, when $x_i \succeq x_j$ with $i \in I, j \in I^c$. Let us relate each t_G with the numerical distributions t_{G_1}, t_{G_2} appearing in the Wick expansions of $T(I)$ and $T(I^c)$,

$$T(I) = \sum_{f_1,\bar{f}_1,b_1}\sum_{G_1\in\mathcal{G}^{|I|}_{f_1,\bar{f}_1,b_1}} t_{G_1}(I)W_{G_1}(I) \tag{3.13}$$

$$T(I^c) = \sum_{f_2,\bar{f}_2,b_2}\sum_{G_2\in\mathcal{G}^{|I^c|}_{f_2,\bar{f}_2,b_2}} t_{G_2}(I^c)W_{G_2}(I^c) \tag{3.14}$$

Thus,

$$T_n(X) = T(I)T(I^c) = \sum t_{G_1}t_{G_2}\underbrace{W_{G_1}W_{G_2}}_{\sum_{G'} w_{G'}W_{G'}} = \sum t_G(X)W_G(X). \tag{3.15}$$

Now, let us look at the relation between t_G, t_{G_1} and t_{G_2} that appeared in (3.15),

$$t_G(X) = t_{G_1}(I)t_{G_2}(I^c)w_{G'}(X). \tag{3.16}$$

We need to determine $w_{G'}$ and to this end, we look at the graphs G, G_1 and G_2. The graph G decomposes into two subgraphs G_1 and G_2. A part of the external fermion lines of G_1 are also external lines of G, while the other part is connected to some of the fermion lines of G_2 and turns into an inner fermion line of G. Similarly, a part of the external ϕ-lines of G_1 can be connected to external ϕ-lines of G_2 and vice versa. Let us denote by $i_{\psi\overline{\psi}}$ and

$i_{\phi\phi}$ the number of internal lines in G arising this way between G_1 and G_2 of type $\psi\overline{\psi}$ and $\phi\phi$, respectively. Then,

$$w_{G'}(X) = \prod_{(i,j)\in I\times I^c} w_{\psi\overline{\psi}}(x_i,x_j)^{a_{ij}} w_{\phi\phi}(x_i,x_j)^{b_{ij}}$$

where $w_{\psi\overline{\psi}}$ and $w_{\phi\phi}$ are the two-point functions of $\psi\overline{\psi}$ and $\phi\phi$ and $a_{ij}\in\{0,1\}$ and $b_{ij}\in\mathbb{N}$ are the number of internal lines of type $\psi\overline{\psi}$ and $\phi\phi$ between the vertices i and j, respectively. Then, $\sum_{i,j} a_{ij} = i_{\psi\overline{\psi}}$ and $\sum_{i,j} b_{ij} = i_{\phi\phi}$. Denote by sd_1 and sd_2 the scaling degrees with respect to D_n of t_{G_1} and t_{G_2}, respectively. Then equation (3.16) leads to

$$sd(t_G) \le sd_1 + sd_2 + i_{\psi\overline{\psi}} \tag{3.17}$$

As we have mentioned before, external fermion, or boson, lines from G_1, G_2 can connect to form internal lines in G. So, the number of external is the number of external lines from G_1 and G_2 deducted by the number of internal lines they form, and this is expressed in relation

$$f_1 + f_2 = f + i_{\psi\overline{\psi}}. \tag{3.18}$$

This implies that

$$sd(t_G) \le sd_1 + sd_2 + (f_1 + f_2 - f) \tag{3.19}$$

where f_i, f is the number of external $\psi-$lines of G_i, G, respectively.

The inequality (3.9) along with the hypothesis $sd_1 \le n_1 - f_1$ and $sd_2 \le n_2 - f_2$ implies the inductive step $sd(t_G) \le (n_1+n_2) - f = n - f$. For the base case, i.e. n=1, take $W_1 \in \mathcal{W}$. In this case, $f \le 1$ and $t_G(X) = 1 \Rightarrow sd(t_G) = 0$. Therefore, $n - f \ge 1 - (1) = 0 = sd(t_G)$. □

Definition 28. Let $t \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ be a distribution and $sd(t)$ the scaling degree of t. Then, the superficial degree of divergence is defined as

$$\omega(t) \doteq sd(t) - 2k. \tag{3.20}$$

Fact: Based on Theorem 26, we can obtain a criteria for the uniqueness of distributions by looking at the superficial degree of divergence ω. If $\omega(t) < 0$ there is a unique extension of t. Now, if $0 \le \omega(t) < \infty$ we have the case of finite renormalization.

Corollary 29. Let G be a graph appearing in the Wick expansion of the time-ordered product T_n, with $W_i \in \mathcal{W}$ and $t_G \in \mathcal{D}'(\mathbb{R}^{2(n-1)})$ the numerical distribution associated with G. Then, the superficial degree of divergence of t_G is bounded by

$$\omega(t_G) \le 2 - n - f \tag{3.21}$$

The case of finite renormalization occurs when $f = 0$ and $n = 2$ leading to $\omega(t_G) = 0$. There are only two correlation functions that need to be renormalized and are represented

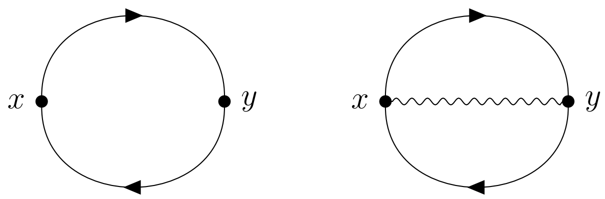


Figure 2 – Graphs representing renormalization cases of VEV $\langle Tj^\mu j^\nu\rangle$ and $\langle T\phi j^\mu \phi j^\nu\rangle$, respectively.

in Figure 2. Notice from the Remark on page 32 that the $\langle Tj^\mu j^\nu\rangle$ is unique by the Ward Identities. Then, the only case left that is open to finite renormalization is $\langle T\phi j^\mu \phi j^\nu\rangle$.

We obtain uniquely defined $T[W_1 \cdots W_n]$ if $\omega(t_G) < 0$, which happens when $f \geq 1$ and $n = 2$ or any value of f and b for $n \geq 3$.

We can proceed in an analogous way and try to obtain the previous results including derivatives of the massive boson. With this in mind we consider the set

$$\mathcal{W}' \doteq \mathcal{W} \cup \{\partial_\mu \phi^r, \partial\!\!\!/ \phi^r \psi, \overline{\psi} \partial\!\!\!/ \phi^r, \partial_\mu \phi j^\mu, \dots\} \tag{3.22}$$

with $r \geq 1$.

We have that the product T_n factorizes similarly to (3.13) and (3.14). The scaling degree increases by 1 for each derivative, therefore $sd(w_{\phi\partial\phi}) = 1$ and $sd(w_{\partial\phi\partial\phi}) = 2$. The bound on $sd(t_G)$ will be found in a similar way as before. This time we face a slightly different case since there are fields with and without derivatives. Thus, $sd(w) \leq i_{\psi\overline{\psi}} + 2i_{bb}$ with i_{bb} being the number of internal bose-bose lines, $bb \in \{\phi\phi, \phi\partial\phi, \partial\phi\partial\phi\}$, in G. A relation similar to (3.9) can be found,

$$b + 2i_{bb} \leq n \tag{3.23}$$

where b is the number of external bose-bose lines of G. This relation applied to $sd(w)$ leads to the inequality

$$sd(w) \leq 2n - f - b. \tag{3.24}$$

We now show that this inequality is the same for t_G.

Proposition 30. Let G be a graph associated with the Wick expansion of the time-ordered product T_n given in Eq.(3.11) with $W_i \in \mathcal{W}'$ and t_G the numerical distribution associated with G. The scaling degree of t_G with respect to D_n is

$$sd(t_G) \leq 2n - f - b. \tag{3.25}$$

Proof. Begin by denoting sd_1 and sd_2 the scaling degrees with respect to D_n of t_{G_1} and t_{G_2}, respectively. Thus, equation (3.17) becomes

$$\begin{aligned} sd(t_G) &\leq sd_1 + sd_2 + 2i_{bb} \\ &= sd_1 + sd_2 + (f_1 + f_2 - f) + (b_1 + b_2 - b) \end{aligned} \tag{3.26}$$

where f_i, f is the number of external $\psi-$lines and b_i, b the number of external bose lines of G_i, G, respectively. The relations

$$f_1 + f_2 = f + i_{\psi\overline{\psi}} \qquad b_1 + b_2 = b + 2i_{bb} \tag{3.27}$$

were used.

To prove the inductive step, consider along with the hypothesis $sd_1 \leq 2n_1 - f_1 - b_1$ and $sd_2 \leq 2n_2 - f_2 - b_2$. It implies that $sd(t_G) \leq 2(n_1 + n_2) - f - b = 2n - f - b$.

Now, for the base case, take $W_1 \in \mathcal{W}'$. There will be only one graph G, so $f \leq 1$, $b \leq 1$ and $t_G(X) = 1 \Rightarrow sd(t_G) = 0$. Thus, $2n - f - b \geq 2 - 1 - 1 = 0 = sd(t_G)$. □

Corollary 31. Let G be a graph associated with the Wick expansion of the time-ordered product T_n, with $W_i \in \mathcal{W}'$ and $t_G \in \mathcal{D}'(\mathbb{R}^{2(n-1)})$ the numerical distribution associated with G. Then, the bound on the superficial degree of divergence is

$$\omega(t_G) \leq 2 - f - b \tag{3.28}$$

The case of finite renormalization, i.e. $\omega(t_G) \geq 0$, occurs when $f = 0$ and $b = 0, 1, 2$ or when $f = 1$, $b = 0, 1$ and or $f = 2$, $b = 0$. Since n is arbitrary, our model is only renormalizable. This means that there is an infinite number of graphs to be renormalized. For $f > 2$ and $b = 0$, $f = 0$ and $b > 2$ or $f = 1$ and $b > 1$ or $f > 1$ and $b = 1$ one obtains $\omega(t_G) < 0$.

3.2.2 Ward Identities

We can introduce one more normalization condition, the well-known Ward Identities[3]. These identities are connected to physical requirements and closely related to the gauge invariance of the theory [38, 63].

Notation: From now on we denote $W_k := W_k(x_k)$. We also denote the dependence of the variable x by omitting the index, i.e. $\psi := \psi(x)$, $\partial_\mu := \frac{\partial}{\partial x^\mu}$, and omit the dots from Wick ordering ::, e.g. $\phi_1\psi_1\overline{\psi}_2\phi_2 \doteq\, :\phi(x_1)\psi(x_1)::\overline{\psi}(x_2)\phi(x_2):$.

We define $q(\psi) := 1$, $q(\overline{\psi}) := -1$ and $q(\phi) := 0$ as convention of the charge[4]. We also define $q(:W_1 \cdots W_n:) \doteq q(W_1) + \cdots + q(W_n)$.

[3] In fact, these are the Ward-Takahashi identities, but for brevity we call them Ward identities.
[4] This coincides with the definition in [57].

Theorem 32 (Ward Identities)**.** Let $W_i \in \mathcal{W}$ be a Wick sub polynomial as defined in (3.6). Then, the vacuum bubbles can be normalized such that

$$\partial_\mu T[j^\mu W_1 \cdots W_n] = -\sum_{l=1}^{n} \delta(x-x_l) q(W_l) T[W_1 \cdots W_n] \tag{3.29}$$

where $q(W_l)$ is the charge of W_l.

Proof. Let us remember that the VEVs of the time-ordered products are fixed, except for the vacuum bubbles. A possible violation of a Ward identity, an anomaly, can only appear in the vacuum sector [57]. We show that the Ward identities are automatically satisfied for all cases except one that needs an appropriate choice of normalization of the vacuum expectation value of the time-ordered product. Hence, we divide this proof in two cases, when $n = 1$ and $n \geq 2$.

Case I: $n = 1$.

In this case, we analyze the anomaly of the form $\langle Tj^\mu W_1\rangle$. The only non-zero vacuum expectation is the one with $W_1 = j_1^\nu$. Given an extension $\langle T_0 j^\mu j_1^\nu\rangle$, we know that $\partial_\mu\langle T_0 j^\mu j_1^\nu\rangle = 0$ outside the origin[5] and that $sd(\langle T_0 j^\mu j_1^\nu\rangle) = 2$. Hence, by Theorem 26, we have that

$$\partial_\mu\langle T_0 j^\mu j_1^\nu\rangle = c^\nu \delta(x-x_1) + c_1\, \partial^\nu \delta(x-x_1). \tag{3.30}$$

The renormalization required to remove the anomaly is only admissible if it has the same symmetries as $\langle T_0 j^\mu j_1^\nu\rangle$. In particular, by Lorentz covariance, the LHS of (3.28) coincides with

$$\partial_\mu\langle T_0 \underbrace{U(\Lambda) j^\mu(x) U(\Lambda)^{-1}}_{(\Lambda^{-1})^\mu{}_\sigma j^\sigma(\Lambda x)} \underbrace{U(\Lambda) j_1^\nu(x_1) U(\Lambda)^{-1}}_{(\Lambda^{-1})^\nu{}_\tau j_1^\tau(\Lambda x_1)}\rangle = (\Lambda^{-1})^\mu{}_\sigma \Lambda^\rho{}_\mu (\Lambda^{-1})^\nu{}_\tau \bar{\partial}_\rho \langle T_0 j^\sigma(\Lambda x) j_1^\tau(\Lambda x_1)\rangle \tag{3.31}$$

where we used that $\dfrac{\partial}{\partial x^\mu} = \Lambda^\rho{}_\mu \dfrac{\partial}{\partial(\bar{x}^\rho)}$ with $\bar{x}^\rho = \Lambda^\rho{}_\mu x^\mu$. On one hand, by using $\Lambda^\mu{}_\sigma(\Lambda^{-1})^\rho{}_\mu = \delta^\rho{}_\sigma$ and (3.28), we have that the RHS of (3.29) becomes

$$\begin{aligned}(\Lambda^{-1})^\nu{}_\tau \bar{\partial}_\sigma \langle T_0 j^\sigma(\Lambda x) j_1^\tau(\Lambda x_1)\rangle &= (\Lambda^{-1})^\nu{}_\tau c^\tau \delta(\Lambda x - \Lambda x_1) + c_1\, (\Lambda^{-1})^\nu{}_\tau \bar{\partial}^\tau \delta(\Lambda x - \Lambda x_1) \\ &= (\Lambda c)^\nu \delta(\Lambda x - \Lambda x_1) + c_1\, \partial^\nu \delta(\Lambda x - \Lambda x_1) \\ &= (\Lambda c)^\nu \delta(x - x_1) + c_1\, \partial^\nu \delta(x - x_1)\end{aligned} \tag{3.32}$$

where in the last line we used $\delta(\Lambda x) = |det(\Lambda)|\delta(x) = \delta(x)$. On the other hand, this should be equal to the right hand side of (3.28), so

$$(\Lambda c)^\nu \delta(x-x_1) + c_1\, \partial^\nu \delta(x-x_1) = c^\nu \delta(x-x_1) + c_1\, \partial^\nu \delta(x-x_1). \tag{3.33}$$

[5] Outside the origin holds $\partial_\mu\langle Tj^\mu j_1^\nu\rangle = \langle T\psi_1\overline{\psi}\rangle(\overleftarrow{\not\partial} + \not\partial)\langle T\psi\overline{\psi}_1\rangle\gamma^\nu = 0$.

To preserve the covariance of the extension T_0, the 4-vector c^ν must also be Lorentz invariant. But the only vector with such characteristic is the null-vector, so $c^\nu = 0$. Then, there remains the following anomaly

$$\partial_\mu \langle T_0 j^\mu j_1^\nu \rangle = c\, \partial^\nu \delta(x - x_1) \tag{3.34}$$

Thus, defining $\langle T j^\mu j_1^\nu \rangle := \langle T_0 j^\mu j_1^\nu \rangle - c\, \eta^{\mu\nu} \delta(x - x_1)$, $\eta^{\mu\nu}$ is the Minkowski metric, we obtain

$$\partial_\mu \langle T j^\mu j_1^\nu \rangle = c\, \partial^\nu \delta(x - x_1) - c\, \partial^\nu \delta(x - x_1) = 0 \tag{3.35}$$

which removes the anomaly and satisfies Eq. (3.29).

Case II: $n \geq 2$

The second and last case left to be analyzed is when $n \geq 2$ and $\omega \leq 0$. The anomaly is given by

$$\partial_\mu \langle T j^\mu W_1 \cdots W_n \rangle - \sum_{k=1}^{n} \delta(x - x_k) q(W_k) \langle T W_1 \cdots W_n \rangle = P(\partial) \delta(x - x_1) \cdots \delta(x - x_n) \tag{3.36}$$

and from [57, Appendix B] we know that $P(\partial)$ has the form

$$P(\partial) = \sum_{k=1}^{n} \partial_\mu^{x_k} P_1^\mu(\partial), \qquad P_1^\mu(\partial) \text{ is a polynomial in } \partial \equiv (\partial_{x_1}, \ldots, \partial_{x_n}). \tag{3.37}$$

In particular, $P(\partial)$ is a differential operator of degree at least one. We have that, by Corollary 29, $T[W_1 \cdots W_n]$ with $W_k \in \mathcal{W}$ are uniquely defined. Then,

$$\begin{aligned} \underbrace{\partial_\mu \underbrace{\langle T j^\mu W_1 \cdots W_n \rangle}_{\omega \leq -1}}_{\omega \leq 0} - \sum_{k=1}^{n} q(W_k) \underbrace{\delta(x - x_k) \underbrace{\langle T W_1 \cdots W_n \rangle}_{\omega \leq 0}}_{\omega \leq 0} &= \sum_{k=1}^{n} \partial_\mu^{x_k} P_1^\mu(\partial) \delta \cdots \delta \\ &= -\underbrace{\partial_\mu P_1^\mu(\partial) \delta \cdots \delta}_{\omega \geq 1} \end{aligned} \tag{3.38}$$

We can look at how the superficial degree of divergence ω of each term of (3.38) is obtained. The first term of the LHS has $n+1$ factors, so $\omega = sd(\langle T j^\mu W_1 \cdots W_n \rangle) - 2(n + 1 - 1) \leq 1 - n - f \leq 1 - 2 - f = -1 - f$, where f is the number of external $\psi-$lines. The derivative increases the scaling degree and ω by one. The second term has n factors in the VEV so $sd(\langle T W_1 \cdots W_n \rangle) \leq n - f$, where f is the number of external $\psi-$lines. We also know that in two dimensions $sd(\delta) = 2$. Then, by definition[6], $\omega \leq sd(\delta) + n - f - 2(n + 1 - 1) \leq 2 - 2 - f = 0 - f$. Thus, this equation can only be satisfied when $P_1^\mu(\partial) = 0$, and we obtain equation (3.29). □

Remark. $\langle T j^\mu j_1^\nu \rangle$ satisfying the VEV of the Ward Identities is unique.

[6] When we take the δ into account, the codimension is calculated with respect to the $n + 1$ variables.

Proof. Let $\langle Tj^\mu j_1^\nu\rangle$ and $\langle T'j^\mu j_1^\nu\rangle$ be two extensions satisfying $\partial_\mu\langle Tj^\mu j_1^\nu\rangle = 0 = \partial_\mu\langle T'j^\mu j_1^\nu\rangle$. As we have seen in the proof of the Theorem, they must differ at most by $c\ \eta^{\mu\nu}\delta(x-x_1)$. Then,

$$\partial_\mu\langle T'j^\mu j_1^\nu\rangle = \partial_\mu\Big(\langle Tj^\mu j_1^\nu\rangle + c\ \eta^{\mu\nu}\delta(x-x_1)\Big) = \partial_\mu\langle Tj^\mu j_1^\nu\rangle + c\ \partial^\nu\delta(x-x_1)$$
$$\Rightarrow c\ \partial^\nu\delta(x-x_1) = 0 \Rightarrow c = 0.$$

□

3.2.3 Extended Ward Identities

Now, our next task is to understand what happens to the time-ordered products when we consider Wick polynomials containing derivatives of ϕ and how these derivatives can be pulled out of the T-product. With this in mind, we will analyze one example, namely $\langle T\partial_\mu\phi_1\phi_2\rangle$. We know that outside the origin

$$\langle T\partial_\mu\phi_1\phi_2\rangle = \left\{\begin{array}{ll}\langle\partial_\mu\phi_1\phi_2\rangle, & x_1 \succeq x_2\\ \langle\phi_2\partial_\mu\phi_1\rangle, & x_2 \succeq x_1\end{array}\right\} = \partial_\mu\langle T\phi_1\phi_2\rangle \tag{3.39}$$

and we must define $\langle T\partial_\mu\phi\phi\rangle \doteq \partial_\mu\langle T\phi\phi\rangle$ across the origin, because

$$\langle T\partial_\mu\phi_1\phi_2\rangle - \partial_\mu\langle T\phi_1\phi_2\rangle = c_0\delta + c_\mu\partial^\mu\delta + \dots$$

where the $sd(\delta) = 2$ and the scaling degree of the VEVs is 1. Therefore, the constants must be zero.

Notation: We remind the reader of the notation given in (2.17) regarding the factorization of Wick polynomials. The derivative acts on the nearest factor, e.g. $\partial XYZ = (\partial X)YZ$ and we also consider $\partial_\mu\phi^r = \partial_\mu(\phi^r)$, and $\partial_\mu\phi^r\psi = (\partial_\mu(\phi^r))\psi$.

The next theorem is dedicated to the formulation of normalization conditions represented in (3.40 - 3.42) and what we call the extended Ward identities (3.43).

Theorem 33. Let $W_k \in \mathcal{W}'$ be a sub polynomial as defined in (3.22). Then, it is possible to define the time-ordered products such that, up to second order[7],

$$\partial_\mu T[\phi^r W_1 \cdots W_n] = T[\partial_\mu \phi^r W_1 \cdots W_n] \tag{3.40}$$

$$\begin{aligned}&(\partial\!\!\!/ + iM)T[:\phi^r\psi: W_1\cdots W_n]\\ &-\sum_{k=1}^{n}(-1)^{\mathrm{f}(W_1\cdots W_{k-1})}\delta(x-x_k)T[W_1\cdots :\phi^r\frac{\partial W_k}{\partial\overline{\psi}}:\cdots W_n] = T[:\partial\!\!\!/\phi^r\psi: W_1\cdots W_n]\end{aligned} \tag{3.41}$$

$$\begin{aligned}&T[W_1\cdots W_n :\overline{\psi}\phi^r:](\overleftarrow{\partial\!\!\!/} - iM)\\ &+\sum_{k=1}^{n}(-1)^{\mathrm{f}(W_{k+1}\cdots W_n)}\delta(x_k-x)T[W_1\cdots :\phi^r\frac{\partial W_k}{\partial\psi}:\cdots W_n] = T[W_1\cdots W_n :\overline{\psi}\partial\!\!\!/\phi^r:]\end{aligned} \tag{3.42}$$

$$\partial_\mu T[:\phi j^\mu: W_1\cdots W_n]+\sum_{k=1}^{n}\delta(x-x_k)q(W_k)T[W_1\cdots :\phi W_k:\cdots W_n] = T[:j^\mu\partial_\mu\phi: W_1\cdots W_n] \tag{3.43}$$

where $r \in \mathbb{N}$, $q(W_k)$ is the charge of W_k, and $\mathrm{f}(W_1 \cdots W_k)$ defined as in (2.17).

Before we prove the Theorem, let us introduce a helpful lemma.

Lemma 34. Let $W_1 \in \{\phi, \phi\psi, \phi\overline{\psi}, \phi j^\mu\}$, $W_2 \in \mathcal{W}'$ and $q(W)$ the charge of W. Then,

$$\begin{aligned}q(W_1)T[W_1W_2] = \sum\Big(&q(W_1''):W_1'W_2':\langle TW_1''W_2''\rangle \quad -:\overline{\psi}_1W_1'W_2':\langle T\frac{\partial W_1''}{\partial\overline{\psi}}W_2''\rangle\\ &+\langle T\frac{\partial W_1''}{\partial\psi}W_2''\rangle:\psi_1W_1'W_2':\Big)\end{aligned}$$

where the sum goes over all possible factorizations.

Proof. We can separate the proof for each W_1 and their possible factorizations.

1. $W_1 = \phi$: $LHS = 0$ since the charge $q(\phi)$ is zero. For the same reason the first term of the RHS is also zero. The other terms of the RHS are zero since there is no ψ or $\overline{\psi}$.

2. $W_1 = \phi\psi$: $LHS = T[:\phi_1\psi_1: W_2]$.

On the RHS: The factorization $W_1' = \phi$, $W_1'' = \psi$ gives us the terms

$$\sum :\phi_1W_2':\langle T\psi_1W_2''\rangle + :\phi_1\psi_1W_2:$$

With the factorization $W_1' = 1$, $W_1'' = \phi\psi$, the RHS becomes

$$\sum\Big(:W_2':\langle T\phi_1\psi_1W_2''\rangle + :\psi_1W_2':\langle T\phi_1W_2''\rangle\Big)$$

[7] By second order we mean the number of test functions g that appear in the expansion of the S-matrix an the interacting version of the fields. Here, it means we need to investigate $T[W_1W_2]$, $T[W_1W_2j^\mu]$, $T[W_1W_2\partial_\mu\phi]$, $T[W_1W_2\psi]$, and $T[W_1 :\phi\psi:]$ with W_1, W_2 a sub polynomial of ∂V or of V.

Both factorizations $W_1' = \phi\psi$, $W_1'' = 1$ and $W_1' = \psi$, $W_1'' = \phi$ yield zero. Therefore, $LHS = RHS$.

3.$W_1 = \phi\overline{\psi}$: This case is analogous to the previous one.

4. $W_1 = \phi j^\mu$: LHS $= 0$. On the RHS: The factorization $W_1' = \phi$, $W_1'' = j^\mu$ gives us

$$\sum \Big(- : \overline{\psi}_1 \phi_1 W_2' : \gamma^\mu \langle T\psi_1 W_2'' \rangle + \langle T\overline{\psi}_1 \gamma^\mu W_2'' \rangle : \psi_1 \phi_1 W_2' : \Big)$$

The factorization $W_1' = 1$, $W_1'' = \phi j^\mu$ gives

$$\sum \Big(- : \overline{\psi}_1 W_2' : \gamma^\mu \langle T\psi_1 \phi_1 W_2'' \rangle + \langle T\overline{\psi}_1 \gamma^\mu \phi_1 W_2'' \rangle : \psi_1 W_2' : \Big)$$

If we consider the other possible factorizations, namely $W_1' = \phi\psi$ and $W_1'' = \overline{\psi}\gamma^\mu$,

$$\sum \Big(q(\overline{\psi}_1) \langle T\overline{\psi}_1 \gamma^\mu W_2'' \rangle : \phi_1 \psi_1 W_2' : - : \phi_1 \overline{\psi}_1 \gamma^\mu \psi_1 W_2 : \Big)$$

$W_1' = \phi\overline{\psi}\gamma^\mu$ and $W_1'' = \psi$,

$$\sum \Big(q(\psi_1) : \phi_1 \overline{\psi}_1 \gamma^\mu W_2' : \langle T\psi_1 W_2'' \rangle + : \phi_1 \overline{\psi}_1 \gamma^\mu \psi_1 W_2 : \Big)$$

$W_1' = \psi$ and $W_1'' = \phi\overline{\psi}\gamma^\mu$,

$$\sum \Big(q(\phi_1 \overline{\psi}_1) \langle T\phi_1 \overline{\psi}_1 \gamma^\mu W_2'' \rangle : \psi_1 W_2' : - : \overline{\psi}_1 \gamma^\mu \psi_1 W_2' : \langle T\phi_1 W_2'' \rangle \Big)$$

$W_1' = \overline{\psi}\gamma^\mu$ and $W_1'' = \phi\psi$, we obtain the following terms

$$\sum \Big(q(\phi_1 \psi_1) : \overline{\psi}_1 \gamma^\mu W_2' : \langle T\phi_1 \psi_1 W_2'' \rangle + : \overline{\psi}_1 \gamma^\mu \psi_1 W_2' : \langle T\phi_1 W_2'' \rangle \Big)$$

The factorization $W_1' = \phi j^\mu$ and $W_1'' = \mathbb{1}$ yields zero. Then, the sum of all contributions to the RHS cancel out and are equal to the $LHS = 0$. □

Proof of Theorem 33. This proof will be divided in three steps. In the first step we show that the T products can be defined such that the vacuum expectation values of (3.40 - 3.43) are satisfied. Note that in (3.41) and (3.42) we have $: \phi^r \dfrac{\partial W_k}{\partial \psi} : \delta$ and not $: \phi^r : \dfrac{\partial W_k}{\partial \psi}\delta$ which would be ill-defined. Let W_1 be a sub polynomial of ∂V or of V and $W_2 \in \{\psi, j^\mu, \partial_\mu \phi\}$.

We wish to show the following equations

$$\partial_\mu \langle T\phi^r W_1 W_2\rangle = \langle T\partial_\mu \phi^r W_1 W_2\rangle \tag{3.44}$$

$$(\not\partial + iM)\langle T:\phi^r\psi : W_1W_2\rangle - \delta(x-x_1)\langle T:\phi^r \frac{\partial W_1}{\partial\overline{\psi}} : W_2\rangle \\ - (-1)^{\mathrm{f}(W_1)}\delta(x-x_2)\langle TW_1 : \phi^r \frac{\partial W_2}{\partial\overline{\psi}} :\rangle = \langle T:\not\partial\phi^r\psi : W_1W_2\rangle \tag{3.45}$$

$$\langle TW_1W_2 : \overline{\psi}\phi^r :\rangle(\overleftarrow{\not\partial} - iM) + \delta(x_2 - x)\langle TW_1 : \phi^r\frac{\partial W_2}{\partial\psi} :\rangle \\ + (-1)^{\mathrm{f}(W_2)}\delta(x-x_1)\langle T:\phi^r\frac{\partial W_1}{\partial\psi} : W_2\rangle = \langle TW_1W_2 : \overline{\psi}\not\partial\phi^r :\rangle \tag{3.46}$$

$$\partial_\mu\langle T:\phi j^\mu : W_1W_2\rangle + \delta(x-x_1)q(W_1)\langle T:\phi W_1 : W_2\rangle \\ + \delta(x-x_2)q(W_2)\langle TW_1 : \phi W_2 :\rangle = \langle T: j^\mu\partial_\mu\phi : W_1W_2\rangle \tag{3.47}$$

To this end, one can take any[8] renormalization of the VEVs on the LHS and then define the RHS by the LHS.

We must show that the construction above is consistent, i.e. the order we pull out the derivatives does not interfere with the result. We take the derivatives out in one specific order and we invite the reader to convince himself in each case that the result is the same no matter the order the derivatives are being taken out of the VEV. The sub polynomials that contain derivatives are $\partial_\mu\phi j^\mu, \not\partial\phi\psi, \overline{\psi}\not\partial\phi$ and $\partial_\mu\phi^\nu$. If we look at charge conservation, the number of cases we need to consider is reduced. The cases we have to analyze are displayed in Appendix A. First, consider case 1a)$\langle T\partial_\mu\phi\partial_\nu\phi_1\rangle$. In this case we simply use (3.44) two times, $\langle T\partial_\mu\phi\partial_\nu\phi_1\rangle = \partial_\mu\partial^1_\nu\langle T\phi\phi_1\rangle$.

Now, take 1b)$\langle T\partial_\mu\phi^r\overline{\psi}_1\not\partial\phi^s_1\psi_2\rangle$.

$$\begin{aligned}\langle T\partial_\mu\phi^r\overline{\psi}_1\not\partial\phi^s_1\psi_2\rangle &= \partial_\mu\langle T\phi^r\overline{\psi}_1\not\partial\phi^s_1\psi_2\rangle \\ &= \partial_\mu\Big(\langle T\phi^r\overline{\psi}_1\phi^s_1\psi_2\rangle(\overleftarrow{\not\partial}^1 - iM) + \delta(x_1-x_2)\langle T\phi^r\phi^s_1\rangle\Big)\end{aligned} \tag{3.48}$$

where we used (3.44) and (3.46).

The next case is 2a)$\langle T\not\partial\phi^r\psi\overline{\psi}_1\not\partial\phi^s_1\rangle$.

$$\begin{aligned}\langle T\not\partial\phi^r\psi\overline{\psi}_1\not\partial\phi^s_1\rangle &= (\not\partial + iM)\langle T\phi^r\psi\overline{\psi}_1\not\partial\phi^s_1\rangle - \delta(x-x_1)\langle T:\phi^r\not\partial\phi^s_1 :\rangle \\ &= (\not\partial + iM)\Big(\langle T\phi^r\psi\overline{\psi}_1\phi^s_1\rangle(\overleftarrow{\not\partial}^1 - iM) + \delta(x-x_1)\langle T:\phi^r\phi^s_1 :\rangle\Big) \\ &= (\not\partial + iM)\langle T\phi^r\psi\overline{\psi}_1\phi^s_1\rangle(\overleftarrow{\not\partial}^1 - iM)\end{aligned} \tag{3.49}$$

where we used (3.45), (3.46), and that $\langle T:\phi^r\phi^s_1 :\rangle = 0$.

[8] It is interesting to remind ourselves that almost all of the VEVs are unique.

Let us look now to 2b)$\langle T\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2\rangle$.

$$
\begin{aligned}
\langle T\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2\rangle &= (\partial\!\!\!/ + iM)\langle T\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2\rangle - \delta(x-x_1)\langle T:\phi^r\partial\!\!\!/\phi_1^s:\partial_\mu\phi_2\rangle \\
&= (\partial\!\!\!/ + iM)\Big(\partial_\mu^2\langle T\phi^r\psi\overline{\psi}_1\phi_1^s\phi_2\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM) + \delta(x_1-x)\partial\!\!\!/^1\partial_\mu^2\langle T:\phi^r\phi_1^s:\phi_2\rangle\Big) \\
&= (\partial\!\!\!/ + iM)\partial_\mu^2\langle T\phi^r\psi\overline{\psi}_1\phi_1^s\phi_2\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM)
\end{aligned}
\tag{3.50}
$$

where we used (3.44 - 3.46) and that $\langle T:\phi^r\phi_1^s:\phi_2\rangle = 0$ for $r,s\geq 1$.

Now, we analyze case 2c)$\langle T\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu\rangle$ as

$$
\begin{aligned}
\langle T\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu\rangle &= (\partial\!\!\!/ + iM)\langle T\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu\rangle - \delta(x-x_1)\langle T:\phi^r\partial\!\!\!/\phi_1^s:j_2^\mu\rangle \\
&+ \delta(x-x_2)\langle T\overline{\psi}_1\partial\!\!\!/\phi_1^s:\phi^r\gamma^\mu\psi_2:\rangle \\
&= (\partial\!\!\!/ + iM)\langle T\phi^r\psi\overline{\psi}_1\phi_1^s j_2^\mu\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM) + (\partial\!\!\!/ + iM)\delta(x_1-x)\langle T:\phi^r\phi_1^s:j_2^\mu\rangle \\
&+ (\partial\!\!\!/ + iM)\delta(x_1-x_2)\langle T\phi^r\psi:\phi_1^s\overline{\psi}_2:\gamma^\mu\rangle \\
&+ \delta(x-x_2)\langle T\overline{\psi}_1\phi_1^s:\phi^r\gamma^\mu\psi_2:\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM) \\
&+ \delta(x-x_2)\delta(x_1-x_2)\langle T:\phi^r\phi_1^s:\rangle\gamma^\mu \\
&= (\partial\!\!\!/ + iM)\langle T\phi^r\psi\overline{\psi}_1\phi_1^s j_2^\mu\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM) \\
&+ \delta(x_1-x_2)(\partial\!\!\!/ + iM)\langle T\phi^r\psi:\phi_1^s\overline{\psi}_2:\gamma^\mu\rangle \\
&+ \delta(x-x_2)\langle T\overline{\psi}_1\phi_1^s:\phi^r\gamma^\mu\psi_2:\rangle(\overleftarrow{\partial\!\!\!/}^1 - iM)
\end{aligned}
\tag{3.51}
$$

where we used equation (3.45) in the first equality and equations (3.44) and (3.46) in the second equality.

Consider case 2d)$\langle\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial_\mu\phi_2^s\rangle$. This is the same as case 1b if we exchange $\overline{\psi}$ and ψ. The result is analogous to the one in case 1b.

Let us look at 3)$\langle T\overline{\psi}\partial\!\!\!/\phi^r\partial_\mu\phi_1^s\psi_2\rangle$.

$$
\begin{aligned}
\langle T\overline{\psi}\partial\!\!\!/\phi^r\partial_\mu\phi_1^s\psi_2\rangle &= \langle T\overline{\psi}\phi^r\partial_\mu\phi_1^s\psi_2\rangle(\overleftarrow{\partial\!\!\!/} - iM) + \delta(x-x_2)\langle T\partial_\mu\phi_1^s\phi^r\rangle \\
&= \partial_\mu^1\langle T\overline{\psi}\phi^r\phi_1^s\psi_2\rangle(\overleftarrow{\partial\!\!\!/} - iM) + \delta(x-x_2)\partial_\mu^1\langle T\phi_1^s\phi^r\rangle
\end{aligned}
\tag{3.52}
$$

where we used (3.44) and (3.46).

Now, consider case 4a)$\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu\rangle$,

$$
\begin{aligned}
\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu\rangle &= \partial_\mu\langle T\phi j^\mu\partial_\nu\phi_1 j_1^\nu\rangle + q(\phi_1 j_1^\nu)\delta(x-x_1)\langle T:\phi\partial_\nu\phi_1 j_1^\nu:\rangle \\
&= \partial_\mu\Big(\partial_\nu^1\langle T\phi j^\mu\phi_1 j_1^\nu\rangle + q(\phi j^\mu)\delta(x_1-x)\langle T:\phi_1\phi j^\mu:\rangle \\
&= \partial_\mu\partial_\nu^1\langle T\phi j^\mu\phi_1 j_1^\nu\rangle
\end{aligned}
\tag{3.53}
$$

where we used (3.47) and that $q(\phi) = q(j^\mu) = 0$.

Now, we look at 4b)$\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda\rangle$,

$$\begin{aligned}\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda\rangle &= \partial_\mu\langle T\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda\rangle + q(\phi_1 j_1^\nu)\delta(x-x_1)\langle T:\phi\partial_\nu\phi_1 j_1^\nu : j_2^\lambda\rangle \\ &+ q(j_2^\lambda)\delta(x-x_2)\langle T\partial_\nu\phi_1 j_1^\nu : \phi j_2^\lambda :\rangle \\ &= \partial_\mu\Big(\partial_\nu^1\langle T\phi j^\mu\phi_1 j_1^\nu j_2^\lambda\rangle + q(\phi j^\mu)\delta(x_1-x)\langle T:\phi_1\phi j^\mu : j_2^\lambda\rangle \\ &+ q(j_2^\lambda)\delta(x_1-x_2)\langle T\phi j^\mu : \phi_1 j_2^\lambda :\rangle\Big) \\ &= \partial_\mu\partial_\nu^1\langle T\phi j^\mu\phi_1 j_1^\nu j_2^\lambda\rangle\end{aligned} \tag{3.54}$$

where we used (3.44) and (3.47).

Consider case 4c)$\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_2^\lambda\rangle$.

$$\begin{aligned}\langle T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_2^\lambda\rangle &= \partial_\mu\langle T\phi j^\mu\partial_\nu\phi_1 j_2^\lambda\rangle + q(\phi_1)\delta(x-x_1)\langle T:\phi\partial_\nu\phi_1 : j_2^\lambda\rangle \\ &+ q(j_2^\lambda)\delta(x-x_2)\langle T\partial_\nu\phi_1 : \phi j_2^\lambda\rangle \\ &= \partial_\mu\partial_\nu^1\langle T\phi j^\mu\phi_1 j_1^\nu j_2^\lambda\rangle\end{aligned} \tag{3.55}$$

where we used (3.47) and (3.44).

And, the last case 4d) $\langle T\partial_\mu\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2\rangle$.

$$\begin{aligned}\langle T\partial_\mu\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2\rangle &= \partial_\mu\langle T\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2\rangle + q(\phi_1^r\overline{\psi}_1)\delta(x-x_1)\langle T:\phi\overline{\psi}_1\not\partial\phi_1^r : \psi_2\rangle \\ &+ q(\psi_2)\delta(x-x_2)\langle T\overline{\psi}_1\not\partial\phi_1^r : \phi\psi_2 :\rangle \\ &= \partial_\mu\Big(\langle T\phi j^\mu\overline{\psi}_1\phi_1^r\psi_2\rangle(\overleftarrow{\not\partial}^1 - iM) + \delta(x-x_1)\langle T:\phi_1^r\phi\overline{\psi}\gamma^\mu : \psi_2\rangle\Big) \\ &+ \delta(x-x_2)\langle T\overline{\psi}_1\phi_1^r : \phi\psi_2 :\rangle(\overleftarrow{\not\partial}^1 - iM) + \delta(x_1-x_2)\delta(x-x_2)\langle T:\phi_1^r\phi :\rangle \\ &= \partial_\mu\langle T\phi j^\mu\phi_1^r\psi_1\psi_2\rangle(\overleftarrow{\not\partial}^1 - iM) + \delta(x-x_2)\langle T\overline{\psi}_1\phi_1^r : \phi\psi_2 :\rangle(\overleftarrow{\not\partial}^1 - iM)\end{aligned} \tag{3.56}$$

where we used (3.46), (3.47), and that $\langle T:\phi^r\psi:\overline{\psi}\rangle = 0$.

The second step is to show (3.40 - 3.43). To this end, we Wick expand (3.40 - 3.43) and use (3.44 - 3.47). First, let us expand the left hand side of (3.40).

$$\partial_\mu T[\phi^r W_1 W_2] = \sum\sum_{\lambda=0}^{r}\binom{r}{\lambda}\Big(\partial_\mu : \phi^\lambda W_1' W_2' : \langle T\phi^{r-\lambda}W_1''W_2''\rangle + : \phi^\lambda W_1' W_2' : \partial_\mu\langle T\phi^{r-\lambda}W_1''W_2''\rangle\Big) \tag{3.57}$$

But the derivative enters the normal ordering and by (3.44) we get

$$\partial_\mu T[\phi^r W_1 W_2] = T[\partial_\mu\phi^r W_1 W_2] \tag{3.58}$$

which is (3.40).

Next, we consider the first term on the LHS of (3.41). We can write it as

$$T[\phi^r\psi W_1 W_2] = \sum\sum_{\lambda=0}^{r}\binom{r}{\lambda}\Big(: \phi^\lambda\psi W_1' W_2' : \langle T\phi^{r-\lambda}W_1''W_2''\rangle + : \phi^\lambda W_1' W_2' : \langle T\phi^{r-\lambda}\psi W_1''W_2''\rangle\Big) \tag{3.59}$$

Using equations (3.44) and (3.45), the fact that the derivative enters the normal ordering, and Dirac equation $(\partial\!\!\!/ + iM)\psi = 0$ gives us

$$
\begin{aligned}
&(\partial\!\!\!/ + iM)T[\phi^r\psi W_1W_2] = \sum\sum_{\lambda=0}^{r-1}\binom{r}{\lambda}\Big(:\partial\!\!\!/\phi^{r-\lambda}\psi W_1'W_2': \langle T\phi^\lambda W_1''W_2''\rangle \\
&+ :\phi^\lambda\gamma^\mu\psi W_1'W_2': \langle T\partial_\mu\phi^{r-\lambda}W_1''W_2''\rangle + :\partial\!\!\!/\phi^{r-\lambda}W_1'W_2': \langle T\phi^\lambda\psi W_1''W_2'\rangle \\
&+ :\phi^\lambda W_1'W_2': \langle T\partial\!\!\!/\phi^{r-\lambda}\psi W_1''W_2''\rangle\Big) + \sum_{\lambda=0}^{r}\binom{r}{\lambda} :\phi^\lambda W_1'W_2': \Big[\delta(x-x_1)\langle T:\phi^{r-\lambda}\frac{\partial W_1''}{\partial\overline{\psi}}: W_2''\rangle \\
&+ (-1)^{\mathrm{f}(W_1'')}\delta(x-x_2)\langle TW_1'':\phi^{r-\lambda}\frac{\partial W_2''}{\partial\overline{\psi}}:\rangle\Big]
\end{aligned}
\tag{3.60}
$$

Observe that the terms where $\lambda = r$ do not contribute to the first sum since $\partial\!\!\!/\phi^{r-r} = 0$. Now, we can use that $\partial\!\!\!/\phi^r = r\phi^{r-1}\partial\!\!\!/\phi$ and $\binom{r}{\lambda} = \frac{r}{r-\lambda}\binom{r-1}{\lambda}$,

$$
\begin{aligned}
&(\partial\!\!\!/ + iM)T[\phi^r\psi W_1W_2] = \sum\sum_{\lambda=0}^{r-1} r\binom{r-1}{\lambda}\Big(:\phi^{r-\lambda-1}\partial\!\!\!/\phi\psi W_1'W_2': \langle T\phi^\lambda W_1''W_2''\rangle \\
&+ :\phi^\lambda\gamma^\mu\psi W_1'W_2': \langle T\phi^{r-\lambda-1}\partial_\mu\phi W_1''W_2''\rangle + :\phi^{r-\lambda-1}\partial\!\!\!/\phi W_1'W_2': \langle T\phi^\lambda\psi W_1''W_2''\rangle \\
&+ :\phi^\lambda\gamma^\mu W_1'W_2': \langle T\phi^{r-\lambda-1}\partial_\mu\phi\psi W_1''W_2''\rangle\Big) + \delta(x-x_1)T[:\phi^r\frac{\partial W_1}{\partial\overline{\psi}}: W_2] \\
&+ (-1)^{\mathrm{f}(W_1)}\delta(x-x_2)T[W_1:\phi^r\frac{\partial W_2}{\partial\overline{\psi}}:]
\end{aligned}
$$

where we used that $T[:\phi^r\frac{\partial W_1}{\partial\overline{\psi}}: W_2] = \sum\sum_{\lambda=0}^{r}\binom{r}{\lambda} :\phi^\lambda W_1'W_2'': \langle T:\phi^{r-\lambda}\frac{\partial W_1''}{\partial\overline{\psi}}: W_2''\rangle$. On the other hand, we know that

$$
\begin{aligned}
T[\partial\!\!\!/\phi^r\psi W_1W_2] &= rT[\phi^{r-1}\partial\!\!\!/\phi\psi W_1W_2] \\
&= \sum\sum_{\lambda=0}^{r-1} r\binom{r-1}{\lambda}\Big(:\phi^\lambda\partial\!\!\!/\phi\psi W_1'W_2': \langle T\phi^{r-1-\lambda}W_1''W_2''\rangle \\
&+ :\phi^\lambda\psi W_1'W_2': \langle T\phi^{r-1-\lambda}\partial\!\!\!/\phi W_1''W_2''\rangle + :\phi^\lambda\partial\!\!\!/\phi W_1'W_2': \langle T\phi^{r-1-\lambda}\psi W_1''W_2''\rangle \\
&+ :\phi^\lambda W_1'W_2': \langle T\phi^{r-1-\lambda}\partial\!\!\!/\phi\psi W_1''W_2''\rangle\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\partial\!\!\!/ + iM)T[\phi^r\psi W_1W_2] &= T[\partial\!\!\!/\phi^r\psi W_1W_2] + \delta(x-x_1)T[:\phi^r\frac{\partial W_1}{\partial\overline{\psi}}: W_2] \\
&+ (-1)^{\mathrm{f}(W_1)}\delta(x-x_2)T[W_1:\phi^r\frac{\partial W_2}{\partial\overline{\psi}}:]
\end{aligned}
\tag{3.61}
$$

which is Eq.(3.41). The steps required to prove equation (3.42) are analogous.

We can proceed to the last case and take the LHS of (3.43) and analyze its parts separately. Let us begin with $T[\phi j^\mu W_1W_2]$,

$$\begin{aligned}\partial_\mu T[\phi j^\mu W_1W_2] = \sum \partial_\mu\Big(&: \phi j^\mu W_1'W_2' : \langle TW_1''W_2''\rangle + : \phi W_1'W_2' : \langle Tj^\mu W_1''W_2''\rangle \\ &+ : j^\mu W_1'W_2' : \langle T\phi W_1''W_2''\rangle + : W_1'W_2' : \langle T\phi j^\mu W_1''W_2''\rangle \\ &+ : \phi\overline{\psi}\gamma^\mu W_1'W_2' : \langle T\psi W_1''W_2''\rangle + \langle T\overline{\psi}\gamma^\mu W_1''W_2''\rangle : \phi\psi W_1'W_2' : \\ &+ : \overline{\psi}\gamma^\mu W_1'W_2' : \langle T\phi\psi W_1''W_2''\rangle + \langle T\phi\overline{\psi}\gamma^\mu W_1''W_2''\rangle : \psi W_1'W_2' : \Big).\end{aligned} \tag{3.62}$$

The derivative acts on the normal ordering and on the VEV. Thus, using (3.44 - 3.47), **(N4)**, Dirac equation, the Ward Identities from Theorem 32, and the current conservation $\partial_\mu j^\mu = 0$, we obtain

$$\begin{aligned}\partial_\mu T[\phi j^\mu W_1W_2] = \sum &: \partial_\mu\phi j^\mu W_1'W_2' : \langle TW_1''W_2''\rangle + : \partial_\mu\phi W_1'W_2' : \langle Tj^\mu W_1''W_2''\rangle \\ &- q(W_1'')\delta(x-x_1) : \phi W_1'W_2' : \langle TW_1''W_2''\rangle - q(W_2'')\delta(x-x_2) : \phi W_1'W_2' : \langle TW_1''W_2''\rangle \\ &+ : j^\mu W_1'W_2' : \langle T\partial_\mu\phi W_1''W_2''\rangle + : W_1'W_2' : \langle T\partial_\mu\phi j^\mu W_1''W_2''\rangle \\ &- q(W_1'')\delta(x-x_1) : W_1'W_2' : \langle T : \phi W_1'' : W_2''\rangle - q(W_2'')\delta(x-x_2) : W_1'W_2' : \langle TW_1'' : \phi W_2'' :\rangle \\ &+ \delta(x-x_1) : \phi\overline{\psi}W_1'W_2' : \langle T\frac{\partial W_1''}{\partial\overline{\psi}}W_2''\rangle + (-1)^{\mathrm{f}(W_1'')}\delta(x-x_2) : \phi\overline{\psi}W_1'W_2' : \langle TW_1''\frac{\partial W_2''}{\partial\overline{\psi}}\rangle \\ &+ : \overline{\psi}\partial\!\!\!/\phi W_1'W_2' : \langle T\psi W_1''W_2''\rangle + \langle T\overline{\psi}W_1''W_2''\rangle : \partial\!\!\!/\phi\psi W_1'W_2' : \\ &- \delta(x_1-x)\langle T\frac{\partial W_1''}{\partial\psi}W_2''\rangle : \phi\psi W_1'W_2' : -(-1)^{\mathrm{f}(W_1'')}\delta(x_2-x)\langle TW_1''\frac{\partial W_2''}{\partial\psi}\rangle : \phi\psi W_1'W_2' : \\ &+ : \overline{\psi}W_1'W_2' : \langle T\partial\!\!\!/\phi\psi W_1''W_2''\rangle + \delta(x-x_1) : \overline{\psi}W_1'W_2' : \langle T : \phi\frac{\partial W_1''}{\partial\overline{\psi}} : W_2''\rangle \\ &+ (-1)^{\mathrm{f}(W_1)}\delta(x-x_2) : \overline{\psi}W_1'W_2' : \langle TW_1'' : \phi\frac{\partial W_2''}{\partial\overline{\psi}} :\rangle + \langle T\overline{\psi}\partial\!\!\!/\phi W_1''W_2''\rangle : \psi W_1'W_2' : \\ &- \delta(x_1-x)\langle T : \phi\frac{\partial W_1''}{\partial\psi} : W_2''\rangle : \psi W_1'W_2' : -(-1)^{\mathrm{f}(W_1)}\delta(x_2-x)\langle TW_1'' : \phi\frac{\partial W_2''}{\partial\psi} :\rangle : \psi W_1'W_2' :\end{aligned} \tag{3.63}$$

Note that all iM-terms cancel out. This can be written as (1) + (2) + (3), where

$$\begin{aligned}(1) = \sum &: \partial_\mu\phi j^\mu W_1'W_2' : \langle TW_1''W_2''\rangle + : \partial_\mu\phi W_1'W_2' : \langle Tj^\mu W_1''W_2''\rangle + : j^\mu W_1'W_2' : \langle T\partial_\mu\phi W_1''W_2''\rangle \\ &+ : W_1'W_2' : \langle T\partial_\mu\phi j^\mu W_1''W_2''\rangle + : \overline{\psi}\partial\!\!\!/\phi W_1'W_2' : \langle T\psi W_1''W_2''\rangle + \langle T\overline{\psi}W_1''W_2''\rangle : \partial\!\!\!/\phi\psi W_1'W_2' : \\ &+ : \overline{\psi}W_1'W_2' : \langle T\partial\!\!\!/\phi\psi W_1''W_2''\rangle + \langle T\overline{\psi}\partial\!\!\!/\phi W_1''W_2''\rangle : \psi W_1'W_2' :\end{aligned}$$

$$\begin{aligned}(2) = -\delta(x_1-x)\Big(\sum &q(W_1'') : \phi W_1'W_2' : \langle TW_1''W_2''\rangle + q(W_1'') : W_1'W_2' : \langle T : \phi W_1'' : W_2''\rangle \\ &- : \phi\overline{\psi}_1W_1'W_2' : \langle T\frac{\partial W_1''}{\partial\overline{\psi}}W_2''\rangle + \langle T\frac{\partial W_1''}{\partial\psi}W_2''\rangle : \phi\psi_1W_1'W_2' : \\ &- : \overline{\psi}_1W_1'W_2' : \langle T : \phi\frac{\partial W_1''}{\partial\overline{\psi}} : W_2''\rangle + \langle T : \phi\frac{\partial W_1''}{\partial\psi} : W_2''\rangle : \psi_1W_1'W_2' : \Big)\end{aligned}$$

$$\begin{aligned}(3) = -\delta(x_2-x)\Big(\sum &q(W_2'') : \phi W_1'W_2' : \langle TW_1''W_2''\rangle + q(W_2'') : W_1'W_2' : \langle T : \phi W_1'' : W_2''\rangle \\ &- (-1)^{\mathrm{f}(W_1)} : \phi\overline{\psi}_2W_1'W_2' : \langle TW_1''\frac{\partial W_2''}{\partial\overline{\psi}}\rangle + (-1)^{\mathrm{f}(W_1)}\langle TW_1''\frac{\partial W_2''}{\partial\psi}\rangle : \phi\psi_2W_1'W_2' : \\ &- (-1)^{\mathrm{f}(W_1)} : \overline{\psi}_2W_1'W_2' : \langle TW_1'' : \phi\frac{\partial W_2''}{\partial\overline{\psi}} :\rangle + (-1)^{\mathrm{f}(W_1)}\langle TW_1'' : \phi\frac{\partial W_2''}{\partial\psi} :\rangle : \psi_2W_1'W_2' : \Big).\end{aligned}$$

We can see that (1) coincides with $T[\partial_\mu \phi j^\mu W_1 W_2]$. By Lemma 34, we can see that (2) coincides with $-\delta(x_1 - x)q(W_1)T[: \phi W_1 : W_2]$, and analogously (3) coincides with $-\delta(x_2 - x)q(W_2)T[W_1 : \phi W_2 :]$. Then,

$$\begin{aligned} \partial_\mu T[\phi j^\mu W_1 W_2] =& T[\partial_\mu \phi j^\mu W_1 W_2] - q(W_1)\delta(x - x_1)T[: \phi W_1 : W_2] \\ & - q(W_2)\delta(x - x_2)T[W_1 : \phi W_2 :] \end{aligned} \tag{3.64}$$

which is exactly (3.43).

Now, our third and last step is to verify that all the axioms of the time-order are still satisfied after our inclusion of derivatives. Note that covariance and the graded symmetry are not changed. We now show that the causal factorization property holds. As we have shown previously, the derivatives can be pulled out irrespective of the order from the VEVs and we obtain a time-ordered product[9] with every $W_i \in \mathcal{W}$. Then, we can use the factorization property for $W_k \in \mathcal{W}$ on the RHS. Since each variable has at most one derivative and the derivatives commute, it is possible to rearrange them in each time-ordered product they should act. We show this property for every case in Appendix B. Here, we present an example and, as the reader may perceive, the other cases can be verified in a similar fashion. Take $T[\partial_\mu \phi j^\mu \partial_\nu \phi_1 j_1^\nu j_2^\lambda]$ and suppose $x \succeq x_1, x_2$. We have that

$$\begin{aligned} T[\partial_\mu \phi j^\mu \partial_\nu \phi_1 j_1^\nu j_2^\lambda] &= \partial_\mu \partial_\nu^1 T[\phi j^\mu \phi_1 j_1^\nu j_2^\lambda] = \partial_\mu \partial_\nu^1 T[\phi j^\mu] T[\phi_1 j_1^\nu j_2^\lambda] \\ &= \partial_\mu T[\phi j^\mu] \partial_\nu^1 T[\phi_1 j_1^\nu j_2^\lambda] \\ &= T[\partial_\mu \phi j^\mu] T[\partial_\nu \phi_1 j_1^\nu j_2^\lambda]. \end{aligned}$$

Therefore, we have that causal factorization is also valid with every $W_k' \in \mathcal{W}'$. □

Remark. In Theorem 33, we can see that up to second order the derivatives can be pulled out in any order yielding

$$\langle TXYZ \rangle = \sum_i c_i P_i \delta_i \langle TX_i Y_i Z_i \rangle \overleftarrow{P}_i$$

where $X, Y, Z \in \mathcal{W}'$ and $X_i, Y_i, Z_i \in \mathcal{W}$, $c_i = \pm 1$, δ_i is a multidimensional delta function or 1, P_i is a differential operator composed of $\{\partial_\mu, (\partial\!\!\!/ + iM), (\overleftarrow{\partial\!\!\!/} - iM)\}$. Here, in our case, the variables of P_i and δ_i are always different. For higher orders, the differential operators might act on the delta distribution and require a more careful analysis[10].

Conjecture: Theorem 33 is valid in all orders, in other words, the Theorem holds for every n.

Theorem 33 implies the following Corollary.

[9] In this time-ordered product act generalized differential operators that are composed of the multiplication of derivatives with deltas.

[10] An example is $\langle T\partial\!\!\!/\phi_1 \psi_1 \overline{\psi}_2 \partial\!\!\!/\phi_2 \phi_3^2 \rangle$. On one hand, if we take first $\partial\!\!\!/^1$ outside we obtain

$$\begin{aligned} \langle T\partial\!\!\!/\phi_1 \psi_1 \overline{\psi}_2 \partial\!\!\!/\phi_2 \phi_3^2 \rangle = (\partial\!\!\!/^1 + iM)\langle T\phi_1 \psi_1 \overline{\psi}_2 \phi_2 \phi_3^2 \rangle (\overleftarrow{\partial\!\!\!/}^2 - iM) - (\partial\!\!\!/^1 + iM)\big(\delta(x_1 - x_2)\langle T : \phi_1 \phi_2 : \phi_3^2 \rangle\big) \\ + \delta(x_1 - x_2)\partial\!\!\!/^2 \langle T : \phi_1 \phi_2 : \phi_3^2 \rangle \end{aligned}$$

Corollary 35. The following normalization conditions are satisfied for all $W_k \in \{V^\mu, \partial.V\}$ up to second order

$$T[\overleftrightarrow{\partial} V W_1 \cdots W_n] = 0 \tag{3.65}$$

$$T[\overleftrightarrow{\partial} V(x) X(y) W_1 \cdots W_n] = 0, \quad X \in \{j^\mu, \partial_\mu \phi\} \tag{3.66}$$

$$T[\overleftrightarrow{\partial} V(x) W_1 \cdots W_n : \phi^r(y)\psi(y) :] = \delta(x-y) T[W_1 \cdots W_n : \phi(x)\phi^r(y)\psi(y) :] \tag{3.67}$$

Here,

$$T[\overleftrightarrow{\partial} V W \cdots W] \doteq T[\partial_\mu V^\mu W \cdots W] - \partial_\mu T[V^\mu W \cdots W]$$

denotes the obstruction to pulling the divergence of V^μ through T.

Notation: We denote

$$T[\partial.V \cdots \partial.V](g^{\otimes n}) = \int d^2x_1 \cdots d^2x_n g(x_1) \cdots g(x_n) T[\partial.V(x_1) \cdots \partial.V(x_n)].$$

Note. After smearing with functions $f, g, g_1, \ldots, g_n$, equation (3.67) yields

$$\begin{aligned} T[\partial V \cdots W_n : \phi^r \psi :](g \otimes \cdots \otimes g_n \otimes f) &= T[V^\mu W_1 \cdots W_n : \phi^r \psi :](-\partial_\mu g) \otimes \cdots \otimes g_n \otimes f) \\ &\quad + T[W_1 \cdots W_n : \phi^{r+1} \psi :](g_1 \otimes \cdots g_n \otimes g f) \end{aligned} \tag{3.68}$$

The last factor is the pointwise product $g.f$.

Remark. In the massive model, both ϕ and $\partial_\mu \phi$ are observable. Such fact is no longer true when we move to the massless case, where only the latter is an observable.

Proposition 36. The only open constant in the model comes from the vacuum bubble. More precisely: Let T, T_0 be two time-orders that satisfy all the normalization conditions, then $\langle T\phi_1 j_1^\mu \phi_2 j_2^\nu \rangle = \langle T_0 \phi_1 j_1^\mu \phi_2 j_2^\nu \rangle + c\, \eta^{\mu\nu} \delta(x_1 - x_2)$.

The freedom in the choice of the constant c in the extension of $\langle T\phi_1 j_1^\mu \phi_2 j_2^\nu \rangle$, plays no role after the adiabatic limit is taken, as we can see in the expressions from (3.74).

Proof. Recall that $T[W_1 \cdots W_n]$, $W_i \in \mathcal{W}'$ are fixed by $T[W_1 \cdots W_n]$, $W_i \in \mathcal{W}$ via (3.40 - 3.43). Now, recall that from Corollary 31 we have that the only VEVs with freedom in the

On the other hand, if we take $\not\partial^2$ outside first yields

$$\begin{aligned} \langle T \not\partial \phi_1 \psi_1 \overline{\psi}_2 \not\partial \phi_2 \phi_3^2 \rangle &= (\not\partial^1 + iM) \langle T \phi_1 \psi_1 \overline{\psi}_2 \phi_2 \phi_3^2 \rangle (\overleftarrow{\not\partial}^2 - iM) + (\not\partial^2 - iM) \big(\delta(x_1 - x_2) \langle T : \phi_1 \phi_2 : \phi_3^2 \rangle \big) \\ &\quad - \delta(x_1 - x_2) \not\partial^1 \langle T : \phi_1 \phi_2 : \phi_3^2 \rangle \end{aligned}$$

The two expressions coincide iff $-\not\partial^1(\delta(x_1 - x_2)t) + \delta(x_1 - x_2)\not\partial^2 t = \not\partial^2(\delta(x_1 - x_2)t) - \delta(x_1 - x_2)\not\partial^1 t$, where $t \doteq \langle T : \phi_1 \phi_2 : \phi_3^2 \rangle$. This is indeed true, and the derivatives can be taken out in both orders yielding the same result.

choice of the extension were $\langle T j_1^\mu j_2^\nu \rangle$ and $\langle T \phi_1 j_1^\mu \phi_2 j_2^\nu \rangle$. The Ward Identities fix $\langle T j_1^\mu j_2^\nu \rangle$. Let $\langle T_0 \phi_1 j_1^\mu \phi_2 j_2^\nu \rangle$ be another extension across the origin, then

$$\langle T \phi_1 j_1^\mu \phi_2 j_2^\nu \rangle - \langle T_0 \phi_1 j_1^\mu \phi_2 j_2^\nu \rangle = c^{\mu\nu} \delta(x_1 - x_2)$$

From symmetry we have that $c^{\mu\nu} = c\,\eta^{\mu\nu}$, with c being a real constant. □

3.3 S-matrix and Interacting Fields

Our final task is to construct Bogoliubov's S-matrix [50], that is a functional defined upon the time-ordered products and is given by the formal series

$$S(g\partial_\mu V^\mu) := \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots dx_n g(x_1) \ldots g(x_n) T[\partial_\mu V^\mu(x_1) \cdots \partial_\mu V^\mu(x_n)] \tag{3.69}$$

where $g \in \mathcal{D}(\mathbb{R}^2)$ is a test function, and $\partial_\mu(\phi j^\mu)$ is the interaction Lagrangian of our model. The physical S-matrix is given by the so-called adiabatic limit where $g(x) \to q$ and $q \in \mathbb{R}$ is fixed. The fixed constant $q \in \mathbb{R}$ is commonly known as the coupling constant. The existence of the adiabatic limit for purely massive theories was proven by Epstein and Glaser in [49]. The problem lies in theories containing massless particles [53]. In this work, we will not address this issue nor the massless limit $m \to 0$.

With Bogoliubov's S-matrix in hand, we can define the interacting fields. Now, let X be a free field. Then, $X\big|_{g\mathcal{L}}$ is the interacting version with interaction $\mathcal{L}(x)$ via Bogoliubov's formula

$$\int d^2x f(x) X\big|_{g\mathcal{L}}(x) \equiv X\big|_{g\mathcal{L}}(f) := -i \frac{d}{d\lambda} S[g\mathcal{L}]^{-1} S[g\mathcal{L} + \lambda f X]\Big|_{\lambda=0}. \tag{3.70}$$

Since our interaction is a divergence, it is expected that the S-matrix is the identity and that the interacting observable fields are equal to the free fields after the adiabatic limit is taken.

Historically, Dirac was the first to propose a formulation of QED containing a gauge invariant electrically charged field [64]. This formulation resembles what we call the dressed Dirac field[11] (3.4). Mandelstam and Steinmann also obtained a similar result while working with gauge invariant fields, but still, this formula needed to be put "by hand" [65, 66]. In the Schroer model, the dressed Dirac field appears as a solution of the classical Euler-Lagrange equations. We present a very interesting formula that connects the interacting Dirac field with the dressed Dirac field. We will call it the "magic formula". As we will see, here it appears naturally. These results are grouped in the following theorem.

Notation: We denote by $X^{(n)}$ the n-th order of the quantity X. For example, $S[gL] = S^{(0)} + S^{(1)} + O(g^2)$. We also denote $g^{\otimes n} = \underbrace{g \otimes \cdots \otimes g}_{n-times}$.

[11] An explanation of this dressing and a way to obtain this dressed Dirac field non-perturbatively can be found in [31].

Definition. Let ϕ be the massive scalar field, ψ the free Dirac field, and g a test function. We call free dressed Dirac field $\psi_g(x) \doteq : e^{ig(x)\phi(x)}\psi(x)$.

Theorem 37. Let $W_k \in \mathcal{W}'$. Suppose that the normalization conditions (3.65-3.67) are satisfied. Thus, we have that, up to second order,

i) The interactions $g\partial_\mu V^\mu$ and $-(\partial_\mu g)V^\mu$ have the same Bogoliubov S-matrix and interacting observables $X \in \{j^\mu, \partial_\mu\phi\}$, that is

$$S[g\partial_\mu V^\mu] = S[-(\partial_\mu g)V^\mu] \tag{3.71}$$

$$X\Big|_{g\partial.V} = X\Big|_{-(\partial g).V} \tag{3.72}$$

ii) The "magic formula" holds

$$\psi\Big|_{g\partial.V} = \psi_g\Big|_{-(\partial g).V} \tag{3.73}$$

We can look at the adiabatic limit of the expressions given in Theorem 37. In [49,67], Epstein and Glaser have shown that for massive theories the adiabatic limit exists in the "strong sense" of operators in a domain in the Fock space. Taking the the adiabatic limit $g \to q$, with q a fixed real constant, $\partial_\mu g \to 0$ and hence the expressions of the theorem become

$$\begin{aligned} S[g\mathcal{L}] &= S[(-\partial g).V] \to S[0] = \mathbb{1} \\ X\Big|_{g\partial V} &= X\Big|_{(-\partial g).V} \to X \\ \psi\Big|_{g\partial V} &= \psi_g\Big|_{(-\partial g).V} \to \psi_q =: e^{iq\phi} : \psi. \end{aligned} \tag{3.74}$$

This is the exact solution of the massive Schroer model presented in Section 3.1.

We see that the freedom in the choice of the constant c in the extension of $\langle T\phi_1 j_1^\mu \phi_2 j_2^\nu\rangle$, discussed in Proposition 36, plays no role after the adiabatic limit is taken, as we can see in the expressions above.

Proof Theorem 37. *i*)The Bogoliubov S-matrix, up to second order, is given by

$$\begin{aligned} S[g\partial.V] &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{\int d^2x_1 \cdots d^2x_n g(x_1)\cdots g(x_n) T[\partial.V(x_1)\cdots\partial.V(x_n)]}_{T[\partial.V\cdots\partial.V](g^{\otimes n})} \\ &= S^{(0)} + S^{(1)} + S^{(2)} + O(g^3) \end{aligned} \tag{3.75}$$

By the normalization condition (3.65), the derivatives can be pulled through the T symbol, yielding[12]

$$S[g\partial.V] = \mathbb{1} + iT[V](-\partial g) - \frac{1}{2}T[VV]((-\partial g)^{\otimes 2}) + O(g^3). \tag{3.76}$$

But this is just $S[(-\partial g).V]$, as claimed.

[12] Here, we used the definition of distributional derivative, $\partial_\mu V^\mu(g) = -V^\mu(\partial_\mu g)$.

Let $X \in \{j^\mu, \partial_\mu\phi$ be an observable. Using Bogoliubov's formula (3.70) and setting the interaction $\mathcal{L} = \partial_\mu V^\mu$, we have that

$$\begin{aligned}\frac{1}{i}\frac{d}{d\lambda}S[g\mathcal{L}+\lambda fX]\Big|_{\lambda=0}^{(\nu)} &= \frac{i^\nu}{\nu!}T[\underbrace{\partial.V\cdots\partial.V}_{\nu-\text{times}}X](g^{\otimes\nu}\otimes f)\\ &= \frac{i^\nu}{\nu!}T[\underbrace{V\cdots V}_{\nu-\text{times}}X]((-\partial g)^{\otimes\nu}\otimes f) = \frac{1}{i}\frac{d}{d\lambda}S[-\partial g.V+\lambda fX]\Big|_{\lambda=0}^{(\nu)}\end{aligned} \tag{3.77}$$

for $\nu = 0, 1, 2$, where we used the normalization condition (3.66). Also, we know that [63],

$$S[g\mathcal{L}]^{-1} = \mathbb{1} - iT[\mathcal{L}](g) + \frac{1}{2}T[\mathcal{L}\mathcal{L}](g^{\otimes 2}) - (\mathcal{L}\mathcal{L})(g^{\otimes 2}) + O(g^3)$$

Thus,

$$\begin{aligned}(S[g\mathcal{L}]^{-1})^{(1)} &= -iT[\partial V](g) = -iT[V](-\partial g) = (S[(-\partial g).V]^{-1})^{(1)}\\ (S[g\mathcal{L}]^{-1})^{(2)} &= \frac{1}{2}T[\partial V\partial V](g^{\otimes 2}) - (\partial V\partial V)(g^{\otimes 2})\\ &= \frac{1}{2}T[VV]((-\partial g)^{\otimes 2}) - VV((-\partial g)^{\otimes 2}) = (S[(-\partial g).V]^{-1})^{(2)}.\end{aligned} \tag{3.78}$$

where we have used (3.65). Thus,

$$(X(f)\big|_{g\partial.V})^{(\nu)} = (X(f)\big|_{-(\partial g).V})^{(\nu)} \tag{3.79}$$

for $\nu = 0, 1, 2$.

ii) Let ψ_g be the dressed free Dirac field. The interacting version of the Dirac field according to Bogoliubov's formula goes

$$\psi(f)\big|_{g\partial.V} = \frac{1}{i}S[g\mathcal{L}]^{-1}\frac{d}{d\lambda}S[g\mathcal{L}+\lambda f\psi]\big|_{\lambda=0} \tag{3.80}$$

and we have that

$$\begin{aligned}\frac{d}{d\lambda}S[g\mathcal{L}+\lambda f\psi]\Big|_{\lambda=0}^{(1)} &= iT[\partial V\psi](g\otimes f) = iT[V\psi]((-\partial g)\otimes f) + iT[:\phi\psi:](gf)\\ &= iT[V\psi]((-\partial g)\otimes f) + i:g\phi\psi:(f)\\ &= \frac{d}{d\lambda}S[(-\partial g).V + \lambda f:e^{ig\phi}:\psi]\Big|_{\lambda=0}^{(1)}\\ &= \frac{d}{d\lambda}S[(-\partial g).V+\lambda f\psi_g]\Big|_{\lambda=0}^{(1)}\end{aligned}$$

where we applied the normalization condition (3.68). The second order term is,

$$\begin{aligned}\frac{1}{i}\frac{d}{d\lambda}S[g\mathcal{L}+\lambda f\psi]\Big|_{\lambda=0}^{(2)} &= -\frac{1}{2}T[\partial V\partial V\psi](g^{\otimes 2}\otimes f)\\ &= -\frac{1}{2}T[VV\psi]((-\partial g)^{\otimes 2}\otimes f) - T[V:\phi\psi:]((-\partial g)\otimes gf) - \frac{1}{2}T[:\phi^2\psi:](g^2f)\\ &= -\frac{1}{2}T[VV\psi]((-\partial g)^{\otimes 2}\otimes f) + iT[V:ig\phi\psi:]((-\partial g)\otimes f) + \frac{(ig)^2}{2}:\phi^2\psi:(f)\\ &= \frac{1}{i}\frac{d}{d\lambda}S[(-\partial g).V+\lambda f:e^{ig\phi}:\psi]\Big|_{\lambda=0}^{(2)}\\ &= \frac{1}{i}\frac{d}{d\lambda}S[(-\partial g).V+\lambda f\psi_g]\Big|_{\lambda=0}^{(2)}.\end{aligned}$$

Here, we have two expansions in the test functions, $g\partial V$ and $e^{ig\phi}$. Combined with equation (3.78), we obtain

$$(\psi(f)\Big|_{g\partial.V})^{(\nu)} = (\psi_g(f)\Big|_{(-\partial g).V})^{(\nu)} \tag{3.81}$$

for $\nu = 0, 1, 2$. This is the desired "magic formula". $\square$

Conclusion

In this thesis, we have perturbatively investigated the Schroer model with a massive boson. Here, we summarize the main achievements of this thesis and point out a few exciting directions for the future. IR divergences were not a problem since they are only present in the massless case. Our model is defined by the interaction $\partial_\mu \phi j^\mu$ and we began our analysis by checking the renormalizability of the T-products and consequently of the model. We verified explicitly that for an interaction ϕj^μ the model is superrenormalizable and after the inclusion of derivatives, i.e. an interaction $\partial_\mu \phi j^\mu$, the model is only renormalizable, that is: there is an infinite number of graphs that permit renormalization and a finite number of open parameters to be fixed. By looking at the superficial degree of divergence we also determined the possible divergent Feynman graphs.

The next step was to check the fulfillment of the Ward identities, known from QED. We have done this in Theorem 32 for every order of perturbation. Since the interaction of our model is a divergence, according to the common physics folklore, we had the following requirements: after the adiabatic limit is taken, 1. Our model should provide a trivial S-matrix, 2. Interacting observable fields should become free fields. We also have one more requirement: 3. Since the "magic formula" holds in first order and it is a rigorous expression of a formula known for many years [64–66], it should hold in every order. These requirements are expressed and discussed in Theorem 37, and are the central result of the work. They also motivate a new set of normalization conditions that we called extended Ward identities. We proved these conditions up to second order in Theorem 33 and conjecture that they hold in every order. These new normalization conditions assure the validity of our requirements and completely determine the open parameters mentioned in the paragraph above, that is, although the model $\partial_\mu \phi j^\mu$ is only renormalizable, all parameters (but one) are completely fixed through these renormalization conditions, as shown in Proposition 36. This open parameter plays no role after the adiabatic limit is taken. The adiabatic limit of our model exists since this is a massive theory, see [49]. If we take the adiabatic limit $g \to q$, our last theorem states that the mentioned features in fact are satisfied, to wit

$$S[g\partial_\mu \phi j^\mu] \to S[0] = \mathbb{1}, \quad X\Big|_{g\partial_\mu \phi j^\mu} \to X, \quad \text{and} \quad \psi\Big|_{g\partial_\mu \phi j^\mu} \to \psi_q =: e^{iq\phi} : \psi. \tag{1}$$

where X is an observable, i.e. $X \in \{j^\mu, \partial_\mu \phi\}$. This shows that the perturbative construction provides the exact solution of the massive Schroer model.

We present now future directions in three steps. The first step is to take the limit to the massless boson, that is, $m \to 0$. It is worth to mention some particularities to this model when $m = 0$. Here, the adiabatic limit only exists in the "weak sense" of correlation functions [53–55], the Wick exponential in $:e^{ig(x)\phi(x)}\psi(x):$ must be replaced by an IR-regularized one, and the electron is an infraparticle [29, 31]. We expect that the (massless) Schroer model can be obtained via the perturbative construction of Epstein-Glaser just as in the present massive case. A second step is to extend the model to $d = 4$ where it coincides with the toy model mentioned in [31] when we replace ϕ, massive or not, by the string-localized escort field.

The final step is to obtain full QED. It has been proposed in [29, 31] to consider a QED interaction $\mathcal{L}(e)$ whose potential is string-localized and lives in a Hilbert space. It differs from the usual Feynman gauge interaction $\mathcal{L}^K$ by a divergence of the type considered in the present work,

$$\mathcal{L}(e) = \mathcal{L}^K + \partial_\mu \phi j^\mu. \tag{2}$$

We conjecture that one can show the analogue of Eq.(3.73) along the lines of the present work. If so, one would have, in the adiabatic limit, the relation

$$\psi\Big|_{q\mathcal{L}(e)} = \psi_q\Big|_{q\mathcal{L}^K}, \tag{3}$$

where ψ_q is the dressed Dirac field defined in (1) with ϕ replaced by the string-localized escort field. This relation has been conjectured in [29, 31], and is a rigorous version, in the Epstein-Glaser scheme, of formulas proposed by Dirac [64], Steinmann [66], and others. It explains the infraparticle nature of the electron, photon cloud superselection, and Gauss' Law, see [31].

Appendix A – Table of Cases - Theorem 33

The tables below display all non-zero VEVs after we consider charge conservation.

Table 1 – Table of VEVs for Eq.(3.44) - $\langle T\partial_\mu\phi^r W_1 W_2\rangle$.

	W_1	W_2
1a)	$\partial_\mu\phi^s$	$\mathbb{1}$
1b)	$\overline{\psi}\partial\!\!\!/\phi^s$	ψ

Table 2 – Table of VEVs for Eq.(3.45) - $\langle T\partial\!\!\!/\phi^q\psi W_1 W_2\rangle$.

	W_1	W_2
2a)	$\overline{\psi}\partial\!\!\!/\phi^r$	$\mathbb{1}$
2b)	$\overline{\psi}\partial\!\!\!/\phi^r$	$\partial_\mu\phi^s$
2c)	$\overline{\psi}\partial\!\!\!/\phi^r$	j^μ
2d)	$\overline{\psi}$	$\partial_\mu\phi^r$

Table 3 – Table of VEVs for Eq.(3.46) - $\langle T W_1 W_2 \overline{\psi}\partial\!\!\!/\phi^r\rangle$.

	W_1	W_2
3)	$\partial_\mu\phi^s$	ψ

Table 4 – Table of VEVs for Eq.(3.47) - $\langle T\partial_\mu\phi j^\mu W_1 W_2\rangle$.

	W_1	W_2
4a)	∂V	$\mathbb{1}$
4b)	∂V	j^ν
4c)	$\partial_\nu\phi^r$	j^ν
4d)	$\overline{\psi}\partial\!\!\!/\phi^r$	ψ

Appendix B – Factorization - Theorem 33

In this appendix we show that all cases analyzed in Theorem 33 satisfy the factorization property with every $W_k \in \mathcal{W}'$. To this end, we will use (3.40 - 3.43).

Case 1a)

$$T[\partial_\mu \phi^r \partial_\nu \phi_1^s] = \partial_\mu T[\phi^r \partial_\nu \phi_1^s] = \partial_\mu \partial_\nu^1 T[\phi^r \phi_1^s] \tag{B.1}$$

Then, suppose $x \succeq x_1$,

$$T[\partial_\mu \phi^r \partial_\nu \phi_1^s] = \partial_\mu \partial_\nu^1 : \phi^r :: \phi_1^s :=: \partial_\mu \phi^r :: \partial_\nu \phi_1^s : \tag{B.2}$$

The case $x_1 \succeq x$ is analogous.

Case 1b)

$$\begin{aligned} T[\partial_\mu \phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] &= \partial_\mu T[\phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] \\ &= \partial_\mu \Big(T[\phi^r \overline{\psi}_1 \phi_1^s \psi_2](\overleftarrow{\partial\!\!\!/}^{\,1} - iM) + \delta(x_1 - x_2) T[\phi^r \phi_1^s] \Big) \end{aligned} \tag{B.3}$$

Consider, for example, the case $x \succeq x_1, x_2$,

$$\begin{aligned} T[\partial_\mu \phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] &= \partial_\mu \Big(: \phi^r : T[\overline{\psi}_1 \phi_1^s \psi_2](\overleftarrow{\partial\!\!\!/}^{\,1} - iM) + \delta(x_1 - x_2) : \phi^r :: \phi_1^s : \Big) \\ &= : \partial_\mu \phi^r : T[\overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] \end{aligned} \tag{B.4}$$

In the other cases $x_1 \succeq x, x_2$ and $x_2 \succeq x, x_1$, we have that $\delta(x_1 - x_2) = 0$. Then, we have respectively,

$$T[\partial_\mu \phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] = : \overline{\psi}_1 \partial\!\!\!/ \phi_1^s : T[\partial_\mu \phi^r \psi_2] \tag{B.5}$$

$$T[\partial_\mu \phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s \psi_2] = - : \psi_2 : T[\partial_\mu \phi^r \overline{\psi}_1 \partial\!\!\!/ \phi_1^s] \tag{B.6}$$

Case 2a)

$$\begin{aligned} T[\partial\!\!\!/ \phi^r \psi \overline{\psi}_1 \partial\!\!\!/ \phi_1^s] &= (\partial\!\!\!/ + iM) T[\phi^r \psi \overline{\psi}_1 \partial\!\!\!/ \phi_1^s] - \delta(x - x_1) T[: \phi^r \partial\!\!\!/ \phi_1^s :] \\ &= (\partial\!\!\!/ + iM) \Big(T[\phi^r \psi \overline{\psi}_1 \phi_1^s](\overleftarrow{\partial\!\!\!/}^{\,1} - iM) + \delta(x - x_1) T[: \phi^r \phi_1^s :] \Big) \\ &\quad - \delta(x - x_1) \partial\!\!\!/^1 T[: \phi^r \phi_1^s :] \end{aligned} \tag{B.7}$$

Suppose $x \succeq x_1$. In this case, $\delta(x - x_1) = 0$. Then, (B.7) becomes

$$\begin{aligned} T[\partial\!\!\!/ \phi^r \psi \overline{\psi}_1 \partial\!\!\!/ \phi_1^s] &= (\partial\!\!\!/ + iM) : \phi^r \psi :: \overline{\psi}_1 \phi_1^s : (\overleftarrow{\partial\!\!\!/}^{\,1} - iM) \\ &= : \partial\!\!\!/ \phi \psi :: \overline{\psi}_1 \partial\!\!\!/ \phi_1 : \end{aligned} \tag{B.8}$$

The other case $x_1 \succeq x$ is analogous.

Case 2b)

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2] &= (\partial\!\!\!/+iM)T[\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2] - \delta(x-x_1)T[:\phi^r\partial\!\!\!/\phi_1^s:\partial_\mu\phi_2]\\ &= (\partial\!\!\!/+iM)\Big(\partial_\mu^2 T[\phi^r\psi\overline{\psi}_1\phi_1^s\phi_2](\overleftarrow{\partial\!\!\!/}^1 - iM) + \delta(x_1-x)\partial\!\!\!/^1\partial_\mu^2 T[:\phi^r\phi_1^s:\phi_2]\Big)\\ &- \delta(x-x_1)\partial\!\!\!/^1\partial_\mu^2 T[:\phi^r\phi_1^s:\phi_2]\end{aligned} \tag{B.9}$$

Suppose $x \succeq x_1, x_2$. Thus,

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2] &= (\partial\!\!\!/+iM)\partial_\mu^2 :\phi^r\psi: T[\overline{\psi}_1\phi_1^s\phi_2](\overleftarrow{\partial\!\!\!/}^1 - iM)\\ &= :\partial\!\!\!/\phi^r\psi: T[\partial\!\!\!/\overline{\psi}_1\phi_1^s\partial_\mu\phi_2]\end{aligned} \tag{B.10}$$

The case $x_1 \succeq x, x_2$ is analogous. Now, consider the case $x_2 \succeq x, x_1$. Then, (B.9) becomes

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s\partial_\mu\phi_2] &= \partial_\mu^2 :\phi_2: \Big[(\partial\!\!\!/+iM)\Big(T[\phi^r\psi\overline{\psi}_1\phi_1^s](\overleftarrow{\partial\!\!\!/}^1 - iM) + \delta(x_1-x)\partial\!\!\!/^1 :\phi^r\phi_1^s:\Big)\\ &- \delta(x-x_1)\partial\!\!\!/^1 :\phi^r\phi_1^s:\Big]\\ &=: \partial_\mu\phi_2: \Big[(\partial\!\!\!/+iM)T[\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s] - \delta(x-x_1):\phi^r\partial\!\!\!/\phi_1^s:\Big]\\ &= :\partial_\mu\phi_2: T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s]\end{aligned} \tag{B.11}$$

Case 2c)

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu] &= (\partial\!\!\!/+iM)T[\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu] - \delta(x-x_1)T[:\phi^r\partial\!\!\!/\phi_1^s:j_2^\mu]\\ &+ \delta(x-x_2)T[\overline{\psi}_1\partial\!\!\!/\phi_1^s:\phi^r\gamma^\mu\psi_2:]\\ &= (\partial\!\!\!/+iM)\Big(T[\phi^r\psi\overline{\psi}_1\phi_1^s j_2^\mu](\overleftarrow{\partial\!\!\!/}^1 - iM) + \delta(x_1-x)T[:\phi^r\phi_1^s:j_2^\mu]\\ &+ \delta(x_1-x_2)T[\phi^r\psi:\phi_1^s\overline{\psi}_2:\gamma^\mu]\Big) - \delta(x-x_1)\partial\!\!\!/^1 T[:\phi^r\phi_1^s:j_2^\mu]\\ &+ \delta(x-x_2)T[\overline{\psi}_1\phi_1^s:\phi^r\gamma^\mu\psi_2:](\overleftarrow{\partial\!\!\!/}^1 - iM)\\ &+ \delta(x-x_2)\delta(x_1-x_2)T[:\phi^r\phi_1^s:\gamma^\mu]\end{aligned} \tag{B.12}$$

Consider the case $x \succeq x_1, x_2$. Then,

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu] &= (\partial\!\!\!/+iM)\Big(:\phi^r\psi: T[\overline{\psi}_1\phi_1^s j_2^\mu](\overleftarrow{\partial\!\!\!/}^1 - iM) + \delta(x_1-x_2):\phi^r\psi::\phi_1^s\overline{\psi}_2:\gamma^\mu\Big)\\ &= :\partial\!\!\!/\phi^r\psi: T[\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu]\end{aligned} \tag{B.13}$$

Now, consider $x_1 \succeq x, x_2$. Then,

$$\begin{aligned}T[\partial\!\!\!/\phi^r\psi\overline{\psi}_1\partial\!\!\!/\phi_1^s j_2^\mu] &= (\partial\!\!\!/+iM):\overline{\psi}_1\phi_1^s: T[\phi^r\psi j_2^\mu](\overleftarrow{\partial\!\!\!/}^1 - iM)\\ &+ \delta(x-x_2):\overline{\psi}_1\phi_1^s::\phi^r\gamma^\mu\psi_2:(\overleftarrow{\partial\!\!\!/}^1 - iM)\\ &= :\overline{\psi}_1\partial\!\!\!/\phi_1^s: T[\partial\!\!\!/\phi^r\psi j_2^\mu]\end{aligned} \tag{B.14}$$

Consider $x_2 \succeq x, x_1$. Thus,

$$\begin{aligned} T[\not{\partial}\phi^r\psi\overline{\psi}_1\not{\partial}\phi_1^s j_2^\mu] &= j_2^\mu(\not{\partial}+iM)\Big(T[\phi^r\psi\overline{\psi}_1\phi_1^s](\overleftarrow{\not{\partial}}^1-iM)+\delta(x_1-x):\phi^r\phi_1^s:\Big) \\ &\quad -\delta(x-x_1)j_2^\mu\not{\partial}^1:\phi^r\phi_1^s: \\ &= j_2^\mu(\not{\partial}+iM)T[\phi^r\psi\overline{\psi}_1\not{\partial}\phi_1^s]-\delta(x-x_1)j_2^\mu:\phi^r\not{\partial}\phi_1^s: \\ &= \ j_2^\mu T[\not{\partial}\phi^r\psi\overline{\psi}_1\not{\partial}\phi_1^s] \end{aligned} \tag{B.15}$$

Case 2d) Analogous to case 1b.

Case 3)

$$\begin{aligned} T[\overline{\psi}\not{\partial}\phi^r\partial_\mu\phi_1^s\psi_2] &= T[\overline{\psi}\phi^r\partial_\mu\phi_1^s\psi_2](\overleftarrow{\not{\partial}}-iM)+\delta(x-x_2)T[\partial_\mu\phi_1^s\phi^r] \\ &= \partial_\mu^1 T[\overline{\psi}\phi^r\phi_1^s\psi_2](\overleftarrow{\not{\partial}}-iM)+\delta(x-x_2)\partial_\mu^1 T[\phi_1^s\phi^r] \end{aligned} \tag{B.16}$$

Consider the case $x \succeq x_1, x_2$. Thus,

$$\begin{aligned} T[\overline{\psi}\not{\partial}\phi^r\partial_\mu\phi_1^s\psi_2] &= \partial_\mu^1:\overline{\psi}\phi^r:T[\phi_1^s\psi_2](\overleftarrow{\not{\partial}}-iM) \\ &= :\overline{\psi}\not{\partial}\phi^r:T[\partial_\mu\phi_1^s\psi_2] \end{aligned} \tag{B.17}$$

The case $x_1 \succeq x, x_2$ becomes

$$\begin{aligned} T[\overline{\psi}\not{\partial}\phi^r\partial_\mu\phi_1^s\psi_2] &= \partial_\mu^1:\phi_1^s:T[\overline{\psi}\phi^r\psi_2](\overleftarrow{\not{\partial}}-iM)+\delta(x-x_2)\partial_\mu^1:\phi_1^s::\phi^r: \\ &=:\partial_\mu^1\phi_1^s:T[\overline{\psi}\not{\partial}\phi^r\psi_2] \end{aligned} \tag{B.18}$$

Now, consider the case $x_2 \succeq x, x_1$. Then,

$$\begin{aligned} T[\overline{\psi}\not{\partial}\phi^r\partial_\mu\phi_1^s\psi_2] &= -\psi_2\partial_\mu^1 T[\overline{\psi}\phi^r\phi_1^s](\overleftarrow{\not{\partial}}-iM) \\ &= -\psi_2 T[\overline{\psi}\not{\partial}\phi^r\partial_\mu\phi_1^s] \end{aligned} \tag{B.19}$$

Case 4a)

$$\begin{aligned} T[\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu] &= \partial_\mu T[\phi j^\mu\partial_\nu\phi_1 j_1^\nu]+q(\phi_1 j_1^\nu)\delta(x-x_1)T[:\phi\partial_\nu\phi_1 j_1^\nu:] \\ &= \partial_\mu\Big(\partial_\nu^1 T[\phi j^\mu\phi_1 j_1^\nu]+q(\phi j^\mu)\delta(x_1-x)T[:\phi_1\phi j^\mu:]\Big) \\ &= \partial_\mu\partial_\nu^1 T[\phi j^\mu\phi_1 j_1^\nu] \end{aligned} \tag{B.20}$$

Consider $x \succeq x_1$. Therefore,

$$T[\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu] = \partial_\mu\partial_\nu^1:\phi j^\mu::\phi_1 j_1^\nu:=:\partial_\mu\phi j^\mu::\partial_\nu\phi_1 j_1^\nu: \tag{B.21}$$

We proceed analogously for $x_1 \succeq x$.

Case 4b)

$$\begin{aligned} T[\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda] &= \partial_\mu T[\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda]+q(\phi_1 j_1^\nu)\delta(x-x_1)T[:\phi\partial_\nu\phi_1 j_1^\nu:j_2^\lambda] \\ &\quad +q(j_2^\lambda)\delta(x-x_2)T[\partial_\nu\phi_1 j_1^\nu:\phi j_2^\lambda:] \\ &= \partial_\mu\Big(\partial_\nu^1 T[\phi j^\mu\phi_1 j_1^\nu j_2^\lambda]+q(\phi j^\mu)\delta(x_1-x)T[:\phi_1\phi j^\mu:j_2^\lambda] \\ &\quad +q(j_2^\lambda)\delta(x_1-x_2)T[\phi j^\mu:\phi_1 j_2^\lambda:]\Big) \\ &= \partial_\mu\partial_\nu^1 T[\phi j^\mu\phi_1 j_1^\nu j_2^\lambda] \end{aligned} \tag{B.22}$$

Consider the case $x \succeq x_1, x_2$. Then,

$$T\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_1^\nu j_2^\lambda] = \partial_\mu\partial_\nu^1 : \phi j^\mu : T[\phi_1 j_1^\nu j_2^\lambda] =: \partial_\mu\phi j^\mu : T[\partial_\nu\phi_1 j_1^\nu j_2^\lambda] \tag{B.23}$$

The other cases $x_1 \succeq x, x_2$ and $x_2 \succeq x, x_1$ are analogous.

Case 4c)

$$\begin{aligned} T[\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_2^\lambda] &= \partial_\mu T[\phi j^\mu\partial_\nu\phi_1 j_2^\lambda] + q(\phi_1)\delta(x-x_1)T[:\phi\partial_\nu\phi_1 : j_2^\lambda] \\ &+ q(j_2^\lambda)\delta(x-x_2)T[\partial_\nu\phi_1 : \phi j_2^\lambda] \\ &= \partial_\mu\partial_\nu^1 T[\phi j^\mu\phi_1 j_2^\lambda] \end{aligned} \tag{B.24}$$

Consider the case $x \succeq x_1, x_2$. Thus,

$$T[\partial_\mu\phi j^\mu\partial_\nu\phi_1 j_2^\lambda] = \partial_\mu\partial_\nu^1 : \phi j^\mu : T[\phi_1 j_2^\lambda] = : \partial_\mu\phi j^\mu : T[\partial_\nu\phi_1 j_2^\nu] \tag{B.25}$$

The other cases $x_1 \succeq x, x_2$ and $x_2 \succeq x, x_2$ are analogous.

Case 4d)

$$\begin{aligned} T[\partial_\mu\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2] &= \partial_\mu T[\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2] + q(\phi_1^r\overline{\psi}_1)\delta(x-x_1)T[:\phi\overline{\psi}_1\not\partial\phi_1^r : \psi_2] \\ &+ q(\psi_2)\delta(x-x_2)T[\overline{\psi}_1\not\partial\phi_1^r : \phi\psi_2 :] \\ &= \partial_\mu\Big(T[\phi j^\mu\overline{\psi}_1\phi_1^r\psi_2](\overleftarrow{\not\partial}^1 - iM) + \delta(x_1-x)T[:\phi_1^r\phi\overline{\psi}\gamma^\mu : \psi_2] \\ &- \delta(x_1-x_2)T[\phi j^\mu\phi_1^r]\Big) - \delta(x-x_1)T[:\phi\phi_1^r\overline{\psi}_1 : \psi_2](\overleftarrow{\not\partial}^1 - iM) \\ &- \delta(x-x_2)\delta(x_1-x_2)T[:\phi\phi_1^r :] + \delta(x-x_2)T[\overline{\psi}_1\phi_1^r : \phi\psi_2 :](\overleftarrow{\not\partial}^1 - iM) \\ &+ \delta(x_1-x_2)\delta(x-x_2)T[:\phi_1^r\phi :] \\ &= \partial_\mu\Big(T[\phi j^\mu\overline{\psi}_1\phi_1^r\psi_2](\overleftarrow{\not\partial}^1 - iM) + \delta(x-x_1)T[:\phi_1^r\phi\overline{\psi}\gamma^\mu : \psi_2] \\ &- \delta(x_1-x_2)T[\phi j^\mu\phi_1^r]\Big) - \delta(x-x_1)T[:\phi\phi_1^r\overline{\psi}_1 : \psi_2](\overleftarrow{\not\partial}^1 - iM) \\ &+ \delta(x-x_2)T[\overline{\psi}_1\phi_1^r : \phi\psi_2 :](\overleftarrow{\not\partial}^1 - iM) \end{aligned} \tag{B.26}$$

Consider the case $x \succeq x_1, x_2$. Then,

$$\begin{aligned} T[\partial_\mu\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2] &= \partial_\mu : \phi j^\mu : T[\overline{\psi}_1\phi_1^r\psi_2](\overleftarrow{\not\partial}^1 - iM) - \delta(x_1-x_2)\partial_\mu : \phi j^\mu :: \phi_1^r : \\ &= : \partial_\mu\phi j^\mu : T[\overline{\psi}_1\not\partial\phi_1\psi_2] \end{aligned} \tag{B.27}$$

Now, consider the case $x_1 \succeq x, x_2$. Thus,

$$\begin{aligned} T[\partial_\mu\phi j^\mu\overline{\psi}_1\not\partial\phi_1^r\psi_2] &=: \overline{\psi}_1\phi_1^r : \partial_\mu T[\phi j^\mu\psi_2](\overleftarrow{\not\partial}^1 - iM) + \delta(x-x_2) : \overline{\psi}_1\phi_1^r :: \phi\psi_2 : (\overleftarrow{\not\partial}^1 - iM) \\ &= : \overline{\psi}_1\not\partial\phi_1 : T[\partial_\mu\phi j^\mu\psi_2] \end{aligned} \tag{B.28}$$

The idea is the same for $x_2 \succeq x, x_1$.

Bibliography

1 SCHROER, B. Infrateilchen in der Quantenfeldtheorie. *Fortschr. Phys.*, v. 173, p. 1527, 1963. 7, 9, 13, 14, 30, 31

2 HANNEKE, D.; FOGWELL, S.; GABRIELSE, G. New measurement of the electron magnetic moment and the fine structure constant. *Phys. Rev. Lett.*, American Physical Society, v. 100, p. 120801, Mar 2008. 13

3 MOTT, N. F. The scattering of fast electrons by atomic nuclei. *Proceedings Mathematical Physical and Engineering Sciences*, The Royal Society, v. 124, p. 425–442, 1929. ISSN 1364-5021,1471-2946. Available at: <http://doi.org/10.1098/rspa.1929.0127>. 13

4 MOTT, N. F. On the influence of radiative forces on the scattering of electrons. *Mathematical Proceedings of the Cambridge Philosophical Society*, Cambridge University Press, v. 27, p. 255, 1931. ISSN 0305-0041,1469-8064. 13

5 BLOCH, F.; NORDSIECK, A. Note on the radiation field of the electron. *Phys. Rev.*, American Physical Society, v. 52, p. 54–59, Jul 1937. Available at: <https://link.aps.org/doi/10.1103/PhysRev.52.54>. 13

6 YENNIE, D.; FRAUTSCHI, S.; SUURA, H. The infrared divergence phenomena and high-energy processes. *Annals of Physics*, v. 13, n. 3, p. 379 – 452, 1961. ISSN 0003-4916. Available at: <http://www.sciencedirect.com/science/article/pii/0003491661901518>. 13

7 FIERZ, W. P. M. Zur theorie der emission langwelliger lichtquanten. *Il Nuovo Cimento Series 10*, Società Italiana di Fisica, v. 15, p. 167–188, 1960. ISSN 0029-6341,1827-6121. Available at: <http://doi.org/10.1007/bf02958939>. 13

8 JAUCH, J.; ROHRLICH, F. The infrared divergence. *Helvetica Physica Acta*, v. 27, p. 613–636, 1954. 13

9 SCHROER, B. *A note on Infraparticles and Unparticles*. arXiv, 2008. Available at: <https://arxiv.org/abs/0804.3563>. 13

10 BUCHHOLZ, D. The physical state space of quantum electrodynamics. *Commun. Math. Phys.*, v. 85, p. 49, 1982. 13

11 BUCHHOLZ, D. Gauss' law and the infraparticle problem. *Phys. Lett.*, B 147, p. 331–334, 1986. 13

12 FERRARI, R.; PICASSO, L. E.; STROCCHI, F. Some remarks on local operators in quantum electrodynamics. *Comm. Math. Phys.*, Springer, v. 35, n. 1, p. 25–38, 1974. Available at: <https://projecteuclid.org:443/euclid.cmp/1103859516>. 13

13 BUCHHOLZ, D.; FREDENHAGEN, K. Locality and the structure of particle states. *Commun. Math. Phys.*, v. 84, p. 1–54, 1982. 13

14 MUND, J.; SCHROER, B.; YNGVASON, J. String–localized quantum fields from wigner representations. *Phys. Lett. B*, v. 596, p. 156–162, 2004. 13

15 MUND, J.; SCHROER, B.; YNGVASON, J. String–localized quantum fields and modular localization. *Commun. Math. Phys.*, v. 268, p. 621–672, 2006. 13

16 GRACIA-BONDÍA, J. M.; MUND, J.; VÁRILLY, J. C. The chirality theorem. *Ann. Henri Poincaré*, v. 19, p. 843–874, 03 2018. 13

17 MUND, J.; OLIVEIRA, E. String-localized free vector and tensor potentials for massive particles with any spin: I. bosons. *Communications in Mathematical Physics*, 09 2016. 13

18 CHEN, T.; FRöHLICH, J.; PIZZO, A. Infraparticle scattering states in non-relativistic qed: I. the bloch-nordsieck paradigm. *Communications in Mathematical Physics*, Springer, v. 294, p. 761–825, 2010. ISSN 0010-3616,1432-0916. Available at: <http://doi.org/10.1007/s00220-009-0950-x>. 13

19 CHEN, T.; FRöLICH, J.; PIZZO, A. Infraparticle scattering states in nonrelativistic quantum electrodynamics. ii. mass shell properties. *Journal of Mathematical Physics*, American Institute of Physics, v. 50, p. 012103, 2009. ISSN 0022-2488,1089-7658. Available at: <http://doi.org/10.1063/1.3000088>. 13

20 HERDEGEN, A. Infraparticle problem, asymptotic fields and haagruelle theory. *Annales Henri Poincaré*, Springer, v. 15, p. 345–367, 2014. ISSN 1424-0637,1424-0661. Available at: <http://doi.org/10.1007/s00023-013-0242-z>. 13

21 JäKEL, C. D.; WRESZINSKI, W. F. A criterion to characterize interacting theories in the wightman framework. *Quantum Studies: Mathematics and Foundations*, 2020. ISSN 2196-5609,2196-5617. Available at: <http://doi.org/10.1007/s40509-020-00227-5>. 13

22 DYBALSKI, W.; TANIMOTO, Y. Infraparticles with superselected direction of motion in two-dimensional conformal field theory. *Communications in Mathematical Physics*, Springer, v. 311, p. 457–490, 2012. ISSN 0010-3616,1432-0916. Available at: <http://doi.org/10.1007/s00220-012-1450-y>. 13

23 DYBALSKI W.; PIZZO, A. Coulomb scattering in the massless nelson model i. foundations of two-electron scattering. *Journal of Statistical Physics*, Springer, v. 154, p. 543–587, 2014. ISSN 0022-4715,1572-9613. Available at: <http://doi.org/10.1007/s10955-013-0857-y>. 13

24 DYBALSKI, W.; PIZZO, A. Coulomb scattering in the massless nelson model ii. regularity of ground states. *Reviews in Mathematical Physics*, World Scientific Publishing Company, v. 31, p. 1950010, 2019. ISSN 0129-055X. Available at: <http://doi.org/10.1142/S0129055X19500107>. 13

25 DYBALSKI, W.; PIZZO, A. Coulomb scattering in the massless nelson model iii: Ground state wave functions and non-commutative recurrence relations. *Annales Henri Poincaré*, Springer, v. 19, p. 463–514, 2018. ISSN 1424-0637,1424-0661. Available at: <http://doi.org/10.1007/s00023-017-0642-6>. 13

26 DYBALSKI, W.; PIZZO, A. Coulomb scattering in the massless nelson model iv. atom-electron scattering. arXiv 1902.08799, 2020. 13

27 BEAUD, V.; DYBALSKI, W.; GRAF, G. M. Infraparticle states in the massless nelson model – revisited. arXiv, 2021. Available at: <https://arxiv.org/abs/2105.05723>. 13

28 GASS, C.; REHREN, K.-H.; TIPPNER, F. On the spacetime structure of infrared divergencies in QED. *Lett Math. Phys.*, v. 112, n. 37, 2022. Available at: <https://doi.org/10.1007/s11005-022-01521-6>. 13

29 MUND, J.; REHREN, K.-H.; SCHROER, B. Gauss law and string-localized quantum field theory. *Journal of High Energy Physics*, 01 2020. 13, 54

30 DYBALSKI, W.; MUND, J. Interacting massless infraparticles in 1+1 dimensions. arXiv, 2021. Accepted for publication by CMP. Available at: <https://arxiv.org/abs/2109.02128>. 13

31 MUND, J.; REHREN, K.-H.; SCHROER, B. Infraparticle quantum fields and the formation of photon clouds. *J. High Energ. Phys.*, n. 83, 2022. Available at: <https://doi.org/10.1007/JHEP04(2022)083>. 14, 30, 49, 54

32 STROMINGER, A. *Lectures on the Infrared Structure of Gravity and Gauge Theory*. [S.l.]: Princeton University Press, 2018. ISBN 1400889855, 9781400889853. 14

33 LEMOS, N. A. *Convite à Física Matemática*. First. [S.l.]: Livraria da Física, 2013. ISBN 9788578611927. 15

34 REED, M.; SIMON, B. *I: Functional Analysis*. [S.l.]: Elsevier Science, 1981. (Methods of Modern Mathematical Physics). ISBN 9780080570488. 15

35 NAKAHARA, M. *Geometry, topology, and physics*. 2nd ed. ed. [S.l.]: Institute of Physics Publishing, 2003. (Graduate student series in physics). ISBN 0750306068,9780750306065. 15

36 JACKSON, J. D. *Classical electrodynamics*. 3rd ed. ed. [S.l.]: Wiley, 1999. ISBN 9780471309321,047130932X. 15

37 REIF, F. *Fundamentals of statistical and thermal physics*. 1. ed. [S.l.]: McGraw-Hill Science/Engineering/Math, 1965. (McGraw-Hill Series in Fundamentals of Physics). ISBN 0070518009,9780070518001. 15

38 ITZYKSON, C.; ZUBER, J.-B. *Quantum Field Theory*. Singapore: McGraw-Hill, 1985. 15, 36

39 O'NEILL, B. *Semi–Riemannian Geometry*. New York: Academic Press, 1983. 17, 18

40 LESCHE, B. *Teoria da Relatividade*. 1. ed. [S.l.]: Editora Livraria da Física, 2005. ISBN 9788588325364. 17

41 RIZZUTI, B. *Relativity Lectures*. 17

42 GAIO, L.; BARROS, D. d.; RIZZUTI, B. Grandezas físicas multidimensionais. *Revista Brasileira de Ensino de Física*, scielo, v. 41, 00 2019. ISSN 1806-1117. 17

43 RIZZUTI, B. F.; GAIO, L. M.; DUARTE, C. Operational approach to the topological structure of the physical space. *Foundations of Science*, Springer, 2020. ISSN 1233-1821,1572-8471. Available at: <http://doi.org/10.1007/s10699-020-09650-8>. 17

44 STREATER, R. F.; WIGHTMAN, A. *PCT, Spin and Statistics, and all that*. New York: W. A. Benjamin Inc., 1964. 19, 20

45 HAAG, R. *Local Quantum Physics*. second edition. Berlin, Heidelberg: Springer, 1996. (Texts and Monographs in Physics). 19

46 STROCCHI, F. *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*. [S.l.]: Oxford University Press, 2013. (International Series of Monographs on Physics). ISBN 9780199671571. 19

47 WIGHTMAN, A. S. How it was learned that quantized fields are operator-valued distributions. *Fortschritte der Physik*, John Wiley and Sons, v. 44, p. 143–178, 1996. ISSN 0015-8208,1521-3978. 19

48 ARAKI, H. Von Neumann algebras of local observables for the free scalar field. *J. Math. Phys.*, v. 5, p. 1–13, 1964. 19

49 EPSTEIN, H.; GLASER, V. The role of locality in perturbation theory. *Annales Poincaré Phys. Theor.*, A 19, p. 211–295, 1973. 23, 24, 25, 27, 49, 50, 53

50 BOGOLIUBOV, N. N.; SHIRKOV, D. V. *Introduction to the Theory of Quantized Fields*. 3rd. ed. New York: Wiley, 1980. 23, 49

51 BRUNETTI, R.; FREDENHAGEN, K. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Commun. Math. Phys.*, v. 208, p. 623–661, 2000. 23, 25, 26, 27

52 BOAS, F.-M. *Gauge Theories in Local Causal Perturbation Theory*. Thesis (PhD) — University of Hamburg, 1999. Available at: <https://arxiv.org/abs/hep-th/0001014>. 23

53 DUCH, P. Weak adiabatic limit in quantum field theories with massless particles. *Annales Henri Poincaré*, Springer, 2018. ISSN 1424-0637,1424-0661. Available at: <http://doi.org/10.1007/s00023-018-0652-z>. 23, 49, 54

54 DUCH, P. Infrared problem in perturbative quantum field theory. *Reviews in Mathematical Physics*, World Scientific, 2021. ISSN 1793-6659. Available at: <https://doi.org/10.1142/S0129055X2150032X>. 23, 54

55 BLANCHARD, P.; SENEOR, R. Green's functions for theories with massless particles. *Ann. Inst. H. Poinc. A*, v. 23, p. 147, 1975. 23, 54

56 CARDOSO, L. T.; MUND, J.; VáRILLY, J. C. String chopping and time-ordered products of linear string-localized quantum fields. *Mathematical Physics, Analysis and Geometry*, Springer Netherlands, v. 21, p. 3, 2018. ISSN 1385-0172,1572-9656. Available at: <http://doi.org/10.1007/s11040-017-9258-9>. 24

57 DÜTSCH, M.; FREDENHAGEN, K. A local (perturbative) construction of observables in gauge theories: The example of QED. *Commun. Math. Phys.*, v. 203, p. 71–105, 1999. 28, 29, 36, 37, 38

58 ABDALLA, E.; ABDALLA, M.; ROTHE, K. *Non-Perturbative Methods in Two-Dimensional Quantum Field Theory*. Singapore: World Scientific, 1991. 30

59 THIRRING, W. E. A soluble relativistic field theory. *Annals of Physics*, Elsevier Science, v. 3, p. 91–112, 1958. ISSN 0003-4916,1096-035X. 30

60 SCHWINGER, J. Gauge invariance and mass. *Physical Review (Series I)*, The American Physical Society, v. 125, p. 397–398, 1962. ISSN 0031-899X,1536-6065. 30

61 SCHWINGER, J. Gauge invariance and mass. ii. *Physical Review (Series I)*, The American Physical Society, v. 128, p. 2425–2429, 1962. ISSN 0031-899X,1536-6065. 30

62 STAMATESCU, K. R. I. Study of a two-dimensional model with axial-current-pseudoscalar derivative interaction. *Annals of Physics*, Elsevier Science, v. 95, p. 202–226, 1975. ISSN 0003-4916,1096-035X. 30

63 SCHARF, G. *Finite Quantum Electrodynamics*. Berlin: Springer, 1995. 36, 51

64 DIRAC, P. A. M. Gauge-invariant formulation of quantum electrodynamics. *Can. J. Phys.*, v. 33, p. 650, 1955. 49, 53, 54

65 MANDELSTAM, S. Quantum electrodynamics without potentials. *Ann. Phys.*, v. 19, p. 1–24, 1962. 49, 53

66 STEINMANN, O. Perturbative QED in terms of gauge invariant fields. *Ann. Phys.*, v. 157, p. 232–254, 1984. 49, 53, 54

67 EPSTEIN, H.; GLASER, V. Adiabatic limit in perturbation theory. In: WIGHTMAN, G. V. A. S. (Ed.). *Renormalization Theory*. 1. ed. [S.l.]: Springer Netherlands, 1976, (NATO Advanced Study Institutes Series 23). p. 193 – 254. 50