Universidade Federal de Juiz de Fora
Instituto de Ciências Exatas
Programa de Pós-Graduação em Física

Vahid Nikoofard

Some Considerations About Field Theories in Commutative and Noncommutative Spaces

Juiz de Fora
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Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Juiz de Fora, na área de concentração em Teoria Quântica de Campos, como requisito parcial para obtenção do título de Doutor em Física.

Orientador: Everton M. C. Abreu

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Ficha catalográfica elaborada através do Modelo Latex do CDC da UFJF
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Nikoofard, Vahid.
Some Considerations About Field Theories in Commutative and Non-commutative Spaces / Vahid Nikoofard. – 2015.
?? f.

Orientador: Everton M. C. Abreu
Tese (Doutorado) – Universidade Federal de Juiz de Fora, Instituto de Ciências Exatas. Programa de Pós-Graduação em Física, 2015.

1. Noncommutative geometry in physics. 2. Soldering formalism and the interference phenomena in quantum field theory. 3. Lagrangian method of quantization. I. Abreu, Everton M. C., orient. II. Título.

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I dedicate this thesis to my dearest wife, Raffaelle Pedroso P. Nikoofard, my darling daughter Sophia and my wonderful parents, Fatemeh Chitsazian and Noorollah Nikoofard. Without them my life is in dark!

Acknowledgements

First and foremost, I would like to thank my loving Creator for making me a curious being who loves to explore His creation and for giving me the opportunity to write this thesis. Without Him, I can do nothing.

Each person who contributed to this piece of work and who made me a bit happier during this period is worth thanking. That includes a lot of people. I will venture to give some names I am fond of. I already apologize for the ones I forgot to mention.

I am very much thankful to my dearest wife, *Raffaelle Pedroso P. Nikoofard*, for her love, understanding, prayers and continuing support to complete this research work. I can not imagine how could be my life without her. She is a woman beyond the imaginations. I am extremely grateful to my wonderful parents, *Fatemeh Chitsazian* and *Noorollah Nikoofard*, for their love, prayers, caring and sacrifices for educating and preparing me for my future. For me they are a symbol of perfect parents and I wish for them all the bests of both worlds. Also I express my special thanks to my darling sisters, *Leila, Narges* and *Maryam* for their support, love and positive energy in my life. They are the bests sisters in the world! I would like to extend my sincerest thanks to my dear aunt, *Dr. Zahra Chitsazian.* She has always been a very loving and caring vice-mother! I would like to thank my great brother-in-laws, *Dr. Mohammadreza Ghiasian* and *Dr. Hossein Nikoofard.* I am so lucky to have all of them in my life!

I would like to thank my advisor, *Prof. Everton M. C. Abreu.* Despite most of the time we were not in the same department, every time I needed him he was present and after every discussion I had a new motivation and his advices helped me to make a significant progress in my work. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly. He gave me the freedom to explore on my own, and at the same time the guidance to recover when my steps faltered. Also I would like to thank him for critically reading this thesis and for his extremely helpful comments. I could not have imagined having a better advisor for my Ph.D study.

Here I can not forget the smiley face of my co-advisor, great master, *Prof. Wilson Oliveira.* He passed away very soon but he is always in our hearts. He was an excellent teacher and wonderful human being. Wilson, we always miss you!

I would also like to thank *Profs. Ilya L. Shapiro, Jorge Ananias Neto, Jose Abdalla Helayël-Neto* and *Mario Junior de Oliveira Neves* for accepting to be members of my thesis jury.

A very special thanks goes out to *Prof. José Abdalla Helayël-Neto*, without his motivation and encouragement I could not able to see the beauty of physics and nature. Prof. Helayel is one of the professors who truly made a difference in my life. He provided me with

direction, continuous support and was always a friend than a mentor. He is one of the best human being that I never ever known! I doubt that I will ever be able to convey my appreciation fully, but I owe him my eternal gratitude.
It was a blessing to have someone as wise and friendly as *Dr. Bruno F. Rizzuti* around. Our conversation helped me to understand many aspects of physics. Beyond the academic relationship, he and his parents, *Odaléa* and *Vicente*, always were present in theses years, very supportive and warm.
I would also like to thank the whole staff of physics department for the nice and informal atmosphere they create, not to mention the interesting and delightful conversions in the *sala de café* and the *churrascos.* In particular, I thank, Profs. *Bernhard Lesche, Maria Luiza Bedran, Vírgilio dos Anjos, Sergio Malker, Maria José V. Bell, Jens Mund, Alexei Deriglazov, Wilson Melo, Sidiney Leonel, Rodrigo Dias, Roberto Pinho, Pablo Zimmermann, José Luiz Valle* and *Fernando Sato.* I have learned many things from them about physics, science in its general aspects, politics and culture.
During my PhD, I have had a very close friendship with *José Amâncio dos Santos, Lucas Cardoso, Sebastião Mauro, Filipe Salles* and *Felipe Pedrosa.* Their infectious enthusiasm and their wide knowledge of their fields got me through many obstacles. They are more than ordinary friends for me and I am very happy to have had the opportunity to know them. Their support and care helped me overcome setbacks and stay focused on my studies. I greatly value their friendship and I deeply appreciate their belief in me.
The institute is an enjoyable place to work and even more, especially when friends and colleagues are around. Thus, I would like to thank *Geraldo, Cléber, Leonarde, Jefferson, Alessandro, both Franciscos (os Chicos), Rodrigo (Paçoca), Thiago, Danilo, Alysson, Maxwell, João Paulo, Humberto, Rafael, Alcides, Mateus, Diogo, Tíberio, Afonso, Renata, Thamiris, Jose Henrique, Erichardson, Dante, Baltazar, Poliane* and *Marcelo*, to name a few.
Many thanks to *Tiago Mendes Rodrigues (Rochedo), Evandro Bastos* and *Adriano Zangirolami* for their support, especially during my early months of my stay in Brazil. I do not know how to thank you Rochedo!
Feeling homesick is a constant obstruction to me. Therefore, I would like to express my gratitude to a number of friends with whom I often feel like at home during my stay in Brazil. In fact, I had the opportunity to have my old Iranian friends together with me in Brazil. We have had many good times talking in Persian together (graças ao plano pré-infinity da TIM). These wonderful friends are *Ali Allahyarzadeh, Masoud Ghanbari, Rouhollah Ebrahimi* and *Dr. Mohammadreza Hadizadeh.*
Of course, more people deserve my acknowledgments. They did not have a direct influence on the content of this thesis, I think most of them don't even care about it, but they certainly had a positive influence of my mental health, without which the last four years would have been much less fun. Among them I would mention my good friends *Rubens*

Melo and his wife *Carolina Baldioti* and their families for hospitality during these years in Juiz de Fora. Also I would like to thank my wife's mother, *Maria José*, her sisters *Estephannie* and *Emmanuelle* and my brother-in-law, *Robson*. A special thank goes out to my great friend *Carlos Meneses* and his wife, *Karina*, for their support and kindness. I am also grateful to *Sr. Domingos Lopes* for all of his assistances. Always kind, helpful and hardworking. Thank you so much!
I also thank dear *Dr. Kate Nixon* for the careful reading and correcting the grammatical errors of this thesis.
I would like to thank the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) for financial support during my PhD studies. In fact, I am indebted to the taxpayers of Brazil for funding my scholarship during these years.

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

– Henri Poincaré

RESUMO

Esta tese é composta por assuntos distintos entre si de teorias quânticas de campos onde alguns deles são descritos em espaços não-comutativos (NC). Em primeiro lugar, analisamos a dinâmica de uma partícula livre sobre uma 2-esfera e através da dinâmica das suas equações de movimento, obtivemos as perturbações NCs neste espaço de fase. Este modelo sugere uma origem para o *Zitterbewegung* do elétron. Depois disso, consideramos uma versão NC da segunda lei de Newton para este modelo, que foi obtido com este cenário geométrico aplicado a este modelo. Em seguida, discutimos um formalismo alternativo relacionado à não-comutatividade chamado DFR onde o parâmetro NC é considerado uma coordenada e demonstramos exatamente que ela tem obrigatoriamente um momento conjugado neste espaço de fase DFR, diferentemente do que alguns autores da atual literatura sobre DFR afirmam. No próximo assunto, usando o formalismo de solda que, em poucas palavras, coloca partículas com quiralidades opostas no mesmo multipleto, soldamos algumas versões NCs de modelos bem conhecidos como modelos de Schwinger quirais e modelos (anti) auto duais no espaço-tempo de Minkowski estendido. Em outro assunto estudado aqui, também construímos a versão NC do modelo de Jackiw-Pi com um grupo de calibre arbitrário e usamos o mapeamento bem conhecido de Seiberg-Witten para obter este modelo NC em termos de variáveis comutativos. Finalmente, utilizamos o formalismo de campos e anticampos (ou método BV) para construir a ação de Batalin-Vilkovisky (BV) do modelo Jackiw-Pi estendido e após o prEntendiocedimento de fixação de calibre chegamos a uma ação completa, pronta para quantização.

Palavras Chaves: Geometria não-comutativa em física. Formalismo de solda e fenômenos de interferência em teoria quântica de campos. Método lagrangiano de quantização.

ABSTRACT

This thesis is composed of distinct aspects of quantum field theories where some of them are described in noncommutative (NC) spaces. Firstly, we have analyzed the dynamics of a free particle over a 2-sphere and through the dynamics of the equations of motion we have derived its NC perturbations in the phase-space. This model suggests an origin for Zitterbewegung feature of the electron. After that we have considered the NC version of Newton's second law for this model, which was obtained with the geometrical scenario applied to this model. Then we have discussed the so-called Doplicher–Fredenhagen–Roberts (DFR) alternative formalism concerning noncommutativity where the NC parameter has a coordinate role and we showed exactly that it has a conjugated momentum in the DFR phase-space, differently of what some authors of the current DFR-literature claims. In the next issue, using the soldering formalism which, in few words, put opposite chiral particles in the same multiplet, we have soldered some NC versions of well known models like the chiral Schwinger model and (anti)self dual models in the extended Minkowski spacetime. Changing the subject, we have constructed the NC spacetime version of Jackiw-Pi model with an arbitrary gauge group and we used the well known Seiberg-Witten map to obtain the NC model expressed in terms of commutative variables. Finally, we have used the field-antifield (or BV method) formalism to construct the Batalin-Vilkovisky (BV) action of the extended Jackiw-Pi model and after the gauge fixing procedure we have arrived at a quantized-ready action for this model.

Key words: Noncommutative geometry in physics. Soldering formalism and the interference phenomena in quantum field theory. Lagrangian method of quantization

LIST OF ABBREVIATIONS AND ACRONYMS

NC	Noncommutative
NCy	Noncommutativity
GR	General Relativity
NCHO	Noncommutative Harmonic Oscillator
QFT	Quantum Field Theory
BV	Batalin-Vilkovisky
JP	Jackiw-Pi
DFR	Doplicher–Fredenhagen–Roberts
QM	Quantum Mechanics
NCQM	Noncommutative Quantum Mechanics
NCFT	Noncommutative Field Theory
WZ	Wess-Zumino
BRST	Becchi–Rouet–Stora–Tyutin
CSM	Chiral Schwinger Model
SW	Seiberg-Witten

Contents

1 Introduction

1.1 Noncommutative physics

The search for the holy grail in theoretical physics is composed of the main challenges that have dwelt among us since the last century. One of these challenges is to unify into a single and consistent framework both theories of quantum mechanics and general relativity. The combination of special relativity and quantum mechanics has already been accomplished through the Klein-Gordon and Dirac approaches. However, the path to reconcile the general relativity with the quantum theory in a completely consistent form is still a mystery.

This so-called quantization procedure of general relativity has stumbled onto another theoretical physics challenge, i.e., the infinities (divergences) that appear in some specific calculations during the quantization process. This issue is directly connected to the understanding of the behavior of quantum fields at the high energy scale which is also connected to the structure of spacetime at (or near) the Planck scale. Understanding the structure of spacetime at this scale is mandatory in order to construct the Hilbert space inner product, essential to the definition of the particle states. There are several formalisms that deal with these questions and one of those is the noncommutative (NC) geometry, which can, for these reasons, be considered as a toy model for quantum gravity.

1.1.1 Noncommutative geometry

The correspondence between geometric spaces and commutative algebras is a well known and basic idea of algebraic geometry. NC geometry generalizes this correspondence to NC algebras. In the physical applications of NC geometry discussed in this work, we are interested in the correspondence between NC algebras of functions on a space and the geometry of the underlying NC space.

The ideas of NC geometry were revived in the 1980's thanks to the works of mathematicians Connes, Drinfel'd and Woronowicz. They generalized the notion of a differential structure to the NC setting [1, 2, 3, 4], i.e. to arbitrary $C^{\star}$-algebras, and also to quantum groups and matrix pseudo-groups. Along with the definition of a generalized integration [5], this led to an operator algebraic description of NC spacetimes - based entirely on the algebras of functions - and it enabled one to define Yang-Mills gauge theories on a large class of NC spaces. Initially, the physical applications were based on geometric interpretations of the standard model and its various fields and coupling constants (the so-called Connes–Lott model) [6, 7, 8]. Gravity was also eventually introduced in a unifying way [9, 10, 11, 12]. Unfortunately this approach suffered from many weaknesses - most glaring was the problem that quantum radiative corrections could not be incorporated in order to give satisfactory predictions - and eventually it died out. Nevertheless, thanks to these mathematicians, the idea of spacetime noncommutativity (NCy) became again very much alive part of theoretical and experimental physics.

1.1.2 The beginning era

It was Heisenberg who suggested, very early, that one could use a NC structure for spacetime coordinates at very small length scales to introduce an effective ultraviolet cutoff. After that, Snyder tackled the idea launched by Heisenberg and published what is considered as the first paper on spacetime NCy in 1947 [13]. Snyder in his seminal work attempted to free us from the infinities that appear in quantum field theory by constructing a five dimensional NC algebra in order to define a minimum length for spacetime structure. Unfortunately, a little time after the Snyder's effort, Yang [14] demonstrated that even in Snyder's NC algebra, the divergences still persisted.
In this approach, Snyder postulated an identity between coordinates and generators of the $SO(4,1)$ algebra. Hence, he promoted the spacetime coordinates to Hermitian operators.

We can construct the Snyder's spacetime algebra conveniently as a modification of the canonical commutation relations of phase-space, given by [15]

$$\begin{aligned} [x^\mu, x^\nu] &= i l_P^2 \hbar^{-1} (x^\mu p^\nu - x^\nu p^\mu) \\ [x^\mu, p_\nu] &= i\hbar \delta^\mu_\nu + i l_P^2 \hbar^{-1} p^\mu p_\nu \\ [p_\mu, p_\nu] &= 0. \end{aligned} \tag{1.1}$$

where we can see the presence of a fundamental minimal length l_P, the scale of NCy. In the limit $l_p \to 0$ we recover the "classical" phase space of quantum mechanics. The commutation relations (1.1) describe a discrete spacetime which, at the same time, respect the Lorentz invariance. However, the original motivation behind these relations was that the introduction of the length scale l_P is analogous to considering hadrons in QFT as extended objects, because at the time renormalization theory was regarded as a distasteful procedure [13, 14, 15]. But, the success of the renormalization method resulted in little attention being paid to the subject for some time.

This result condemned Snyder NCy to be an outcast for more than fifty years until Seiberg and Witten [16] demonstrated that the algebra resulting from string theory embedded into a magnetic field showed itself to have a NC algebra. The so-called Seiberg-Witten (SW) map [16] between commutative and NC gauge theories have explained that gauge symmetries, including diffeomorphisms, can be realized by standard commutative transformations on commutative fields.

1.1.3 String theory with constant background field

String theory, besides loop quantum gravity, is one of the best candidates for quantum gravity. Therefore it has an important role in the study of the fundamental structure of spacetime. String theory has the built-in characteristics of nonlocality and uncertainty of coordinate measurements at short distances. It is the finite mean length of strings, l_s, that necessarily makes the physics nonlocal and forces the shortest length that can be observed by using the strings as probes. Hence, it was not a big surprise when NC spacetime coordinates began to repeatedly emerge from the research concerning string theory. String theory is one of the strongest reasons why spacetime NCy and NC gravitation has been studied so much during the last decade. Seiberg and Witten developed the idea by elegantly proving that when the end point of an open string is constrained to move on D-branes in the presence of a constant (supergravity) B-field background and also the theory is taken in a certain low-energy limit; then the full dynamics of the theory is described by a (supersymmetric) gauge theory on a NC spacetime [16]. In this low-energy limit (Seiberg-Witten limit), the open string modes completely decouple from the closed string modes and only the end point degrees of freedom for the open strings are left to live on a NC spacetime defined by the coordinate commutation relations (1.1). Thus NC gauge theory emerges as a low-energy limit of open string theory with constant antisymmetric background field. Because the closed string modes decouple in this limit, the resulting gauge theories do not have graviton- the quantum of gravitation. Nevertheless, NC gravitation can be studied in the Seiberg-Witten limit by considering first order corrections for the closed string modes. This approach has already provided us important information about NC gravitation and twisted symmetries. In string theory, gravitational interactions have much richer dynamics than in some other NC deformations of GR — especially than the ones based on the invariance under the naive twisted diffeomorphisms.

1.2 Particle over 2-sphere

Classical mechanics is one of the most enlightening starting points for introducing many distinct mathematical tools such as differential equations, symplectic structures [17] and, in particular, the basic concepts of differential geometry. For example, in [18], the author used a potential motion to construct the corresponding geometric setting. In this way, some notions such as Riemann metric space, Christoffel symbols, parallel transport and covariant derivative were introduced. We extend this idea in this thesis. Instead of treating a potential motion, we will describe a free particle constrained to a curved surface. By constructing its corresponding Lagrangian, we are naturally led to a free motion in a Riemann space. Definitions of metric and Christoffel symbols appear in the course of constructing the dynamics of the model.

We will analyze in details the movement of a particle over a 2-sphere, which is the analog to the nonlinear sigma model problem, which was intensely studied in the past (see [19, 20, 21] and references within). Solution of the equations of motion are given in two different ways. Firstly, we will explore the geometrical properties of the model and after that, we will use the Noether charges to decouple the equations of motion. Moreover, due to the symmetrical structure of the 2-sphere, we will establish the equivalence between the motion in a central field and the free particle over the 2-sphere. It turns out that the central potential is proportional to the curvature of the surface. Then, constrained systems may be also a suitable analogue formalism to introduce general relativity, once Einstein interpreted gravity as a deformation of space-time due to the presence of mass [22]. We will also treat the corresponding hamiltonization of the free particle over the 2-sphere according to the Dirac algorithm for constrained systems [23], which enables one to establish the intrinsic relation between the Dirac brackets and Christoffel symbols, since both of them are supposed to provide the proper evolution over the surface where the model is defined, the former in the phase space and the latter, in the configuration space. Although all the calculations are performed classically, we will discuss an application in the quantum realm. We set one possible interpretation of the so-called *Zitterbewegung*, a quivering motion predicted by Schrödinger when he scrutinized the Dirac equation [24]. The time evolution of electron position operators may be separated in two parts: one in a rectilinear movement and the other oscillates in a ellipse as trajectory, resembling the physical variables of a free particle over a 2-sphere. Thus, the *Zitterbewegung* may be interpreted as a position variable constrained to a 2-sphere, if we assume to the electron the structure of a sphere.

In the case of NC classical mechanics, considered here, one can analyze the contribution of NCy in order to add a perturbation in Newton's second law for the systems considered [25]. Namely, since the equations of motions are modified, when treated in a NC space, we can ask about the effects in the acceleration coordinate [25, 26].
The results of this section are published in [27].

1.3 Soldering formalism and interference phenomena

During the last two or three decades of the last century, the fermion-boson mapping was one of the most investigated topics in theoretical physics. The possibility that complicated fermionic actions could be studied through bosonic fields has motivated many physicists at that time. Concerning the chiral bosonization, some importance was given to the fact that in two dimensions we would face anomalous gauge theories in both theories.
At the same time, the study of chiral boson motivated by string theory, instigated another area of research in two dimensional field theory. As a generalization, in supergravity models, the extension of the chiral boson to higher dimensions has naturally introduced the concept of the chiral p-forms. In [28], the authors considered interacting chiral bosons with Abelian and non-Abelian gauge fields. Harada, in [29], investigated the chiral Schwinger model via chiral bosonization and he has analyzed its spectrum. On that time several models were suggested for chiral bosons but latter it was shown that there are some relations between these models [30]. For instance, the Floreanini-Jackiw (FJ) model is the chiral dynamical sector of the more general model proposed by Siegel [31]. The Siegel modes (rightons and

leftons) carry not only chiral dynamics but also symmetry information. The symmetry content of the theory is described by the Siegel algebra, a truncate diffeomorphisms, that disappears at the quantum level.

Studying the deformation of a specific symmetry provides us a better understanding of its structure and also may open new a way to the theories beyond the present ones. In the search for the theories beyond the standard model of particle physics one can investigate the deformation of Lorentz group as the isometry of Minkowski spacetime. The deformation of Lorentz or Poincaré group (κ deformed spacetime) results a NC spacetime and this new structure has some similarities with quantum groups. This NCy is Lie algebraic type according to Hopf algebra classification and recently has been attracted much attention because it is a natural candidate for the spacetime based on which the Doubly Special Relativity has been established. In a recent work the authors through the introduction of a well-defined new proper time have constructed a commutative spacetime that capture all of the characteristics of the NC κ deformed spacetime [32].

In this thesis we have investigated some NC bosonized chiral Schwinger model (CSM) in the extended Minkowski spacetime in the light of the canonical soldering formalism developed in [33]. In the soldering formalism using the iterative Noether procedure one can implement a desired symmetry into a model. The price of this new invariance is inclusion of some new auxiliary fields in the configuration space of the theory. But in the case of two Lagrangians with opposite/complementary symmetries, after doing some iterations one can add up two Lagrangians with the new counterterms and obtain a soldered Lagrangian. In this Lagrangian the auxiliary fields can be removed using their equations of motion. In a few words, after soldering two initial theories with opposite/complementary symmetries we obtain an effective theory that is completely different from the initial ones. This new model has bigger symmetry groups and also is invariant under the desired symmetry. The interesting point is that the final model is not dependent on the initial fields but a new soldered field.

Also we consider the (anti)self-dual models in 3-dimensional κ deformed Minkowski spacetime. These models appears in many occasions in physics, for instance, they are consequence of bosonization of Thirring model in the large mass limit. Self-dual model has a close connection with the Maxwell-Chern-Simons (MCS). An obvious difference between these two models is that, whereas the MCS theory is manifestly gauge-invariant, possessing only first class constraints, the dual model is a purely second-class system.In the usual Minkowski spacetime the soldering of these models yields the Proca model. Hence we expect that the final soldered model be a theory equivalent to the Proca model in this spacetime.

1.4 3-dimensional gauge theories

Three dimensional gauge theories possess theoretical/mathematical interest, in addition they deserve investigation because they describe (1) kinematical processes that are confined to a plane when external structures (magnetic fields, cosmic strings) perpendicular to the plane are present, and (2) static properties of (3 + 1)-dimensional systems in equilibrium with a high temperature heat bath. An important issue is whether the apparently massless gauge theory possesses a mass gap. The suggestion that indeed it does gain support from the observation that the gauge coupling constant squared carries dimension of mass, thereby providing a natural mass-scale (as in the two-dimensional Schwinger model) [34]. Also, without a mass gap, the perturbative expansion is infrared divergent, so if the theory is to have a perturbative definition, infrared divergences must be screened, thereby providing evidence for magnetic screening in the four-dimensional gauge theory at high temperature.

One might study NC theories as interesting analogs of theories of more direct interest, such as Yang-Mills theory. An important point in this regard is that many theories of interest in particle physics are so highly constrained that they are difficult to study. For example, pure Yang-Mills theory with a definite simple gauge group has no dimensionless parameters with which to make a perturbative expansion or otherwise simplify the analysis. From this point of view it is quite interesting to find any sensible and nontrivial variants of these theories. The Chern-Simons expression, when added to the three-dimensional Yang-Mills action, renders the fields massive, while preserving gauge invariance. However, parity symmetry is lost. A trivial way of maintaining parity with this mass generation is through the doublet mechanism. Consider a pair of identical Yang-Mills actions, each supplemented with their own Chern-Simons term, which enters with opposite signs. The parity transformation is defined to include field exchange accompanying coordinate reflection, and this is a symmetry of the doubled theory. Using this method Jackiw and Pi in a seminal paper [35] have offered a theory for massive vector fields, which is gauge invariant and parity preserving. This theory is gauge invariant, but has non-Yang-Mills dynamics. Although formal quantization of the model can be carried out, developing a perturbative calculational method encounters some difficulties.

As this model is non-Abelian, we can not construct its NC counterpart by simply substituting the dot product by the star one and using SW map. Generally in the common method one assumes U(1) as the gauge group [36]. Although it must be mentioned that U(N) is a non-Abelian group but we can analyze it by the common method. But for an arbitrary gauge group the commutation of two gauge transformations is not another gauge transformation of the same group [37]. It will be closed in only the enveloping algebra of the original algebra.

Here we try to construct the NC counterpart of the model proposed by Jackiw and Pi for an arbitrary gauge group using the enveloping algebra of the original algebra. For this reason we have used a method elaborated by J. Wess et al. [37]. The generalization of this method to higher order term of NC parameter can be found in a work done by Ukler et al. [38]. In this thesis we just proceed up to the first order term in our calculations.

1.5 The field-antifield quantization formalism

Batalin-Vilkovisky (BV) or field-antifield formalism [39] is currently the most complete method to deal with quantum gauge field theory. In fact it is a generalization of the BRST formalism [40, 41] that includes the sources of anti-fields into the action. One of the reasons physicist are interested in a BRST invariant action is that it leads to Slavnov-Taylor identities from which one may prove unitarity and renormalizability. Among the various BRST approaches, the BV formalism has the advantage of treating all quantum systems (with/without open algebra's, with/without ghosts for ghosts) in a unified manner. This brings out the essential features more clearly, and that, in turn, might be helpful in quantizing systems, such as the heterotic string or closed-string field theory. In some sense, the BV formalism is a generalization of BRST quantization. In fact, when sources of the BRST transformations are introduced into the configuration space, the BRST approach resembles the field-antifield one[42]. Antifields then, have a simple interpretation: They are the sources for BRST transformations. In this sense, the field-antifield formalism is a general method for dealing with gauge theories within the context of standard field theory.

The general structure of the antibracket formalism is as follows. One introduces an antifield for each field and ghost, thereby doubling the total number of original fields. The antibracket (,) is an odd non-degenerate symplectic form on the space of fields and antifields. The original classical action S_0 is extended to a new action S, in an essentially unique way, to arrive at a theory with manifest BRST symmetry. One equation, the master equation $(S, S) = 0$, reproduces in a compact way the gauge structure of the original theory governed by S_0. Although the master equation resembles the Zinn-Justin equation, the content of both is different since S is a functional of quantum fields and antifields and is a functional of classical fields.

In this thesis we have studies carefully the gauge structure of Jackiw-Pi (JP) model and then we have constructed the corresponding BV action for the U(1)*U(1)*U(1)*SU(N) gauge group. It is obvious that the quantization of this gauge group is possible via BRST approach but we hired BV formalism for having better understanding of the symmetries. Also gauge fixing is simpler in this formalism and moreover the BV action is ready for quantization and study of anomalies.

2 Particle model on 2-sphere and its quantization

2.1 Constrained systems: the basic formalism

The basic path to introduce a constraint into a Lagrangian is via Lagrange multipliers. Equivalently, knowing *a priori* the constraints of the model, one may find one of the variables in terms of the others and include it into the Lagrangian, leading to a new formulation in terms of physical variables, i.e., whose dynamics is independent of the remaining ones. Our first step in these notes is to show the equivalence between the new and former formulations. Besides, we will begin with the notation which will be used here.

Let us consider a free particle constrained to the surface

$$\Phi(x^i) = 0, \tag{2.1}$$

where $x^i = x^i(t);\ i = 1, ..., N$ are the coordinates of the system. There are technical conditions satisfied by the function Φ where we can find one of the variables, say x^1, in terms of the others,

$$\Phi(x^i) = 0 \Leftrightarrow x^1 = f(x^\alpha);\ \alpha = 2, ..., N. \tag{2.2}$$

From now on in this section, Greek letters mean the values $2, ..., N$. In this case, x^1 is a non-physical degree of freedom because its dynamics is dependent of the remaining variables x^α. If $L = L(x^i, \dot{x}^i)$ is the Lagrangian of the free particle in the absence of the constraint (2.1), then the prescription to construct an action in terms of the physical variables x^α is the following,

$$S_1 = \int_{t_1}^{t_2} dt L(x^i, \dot{x}^i)|_{x^1 = f(x^\alpha)}, \tag{2.3}$$

where we have denoted $\dot{x}^i \equiv \frac{dx^i}{dt}$. We can also write that

$$L(x^i, \dot{x}^i)|_{x^1=f(x^\alpha)} = L(x^1 = f(x^\alpha), \dot{x}^1 = \frac{\partial f}{\partial x^\beta}\dot{x}^\beta, x^\alpha, \dot{x}^\alpha) \equiv \bar{L}(x^\alpha, \dot{x}^\alpha). \tag{2.4}$$

The notation $\bar{L}$ indicates the substitution of $x^1 = f(x^\alpha)$ in (2.3) and repeated indexes mean summation, as usual. To obtain the Euler-Lagrange equations of (2.3), we evaluate separately the derivatives of the expression (2.4),

$$\frac{\partial \bar{L}(x^\alpha, \dot{x}^\alpha)}{\partial x^\gamma} = \frac{\partial L(x^i, \dot{x}^i)}{\partial x^1}\bigg|\frac{\partial f}{\partial x^\gamma} + \frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^1}\bigg|\frac{\partial^2 f}{\partial x^\gamma \partial x^\beta}\dot{x}^\beta + \frac{\partial L(x^i, \dot{x}^i)}{\partial x^\gamma}\bigg|, \tag{2.5}$$

$$\frac{\partial \bar{L}(x^\alpha, \dot{x}^\alpha)}{\partial \dot{x}^\gamma} = \frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^1}\bigg|\frac{\partial f}{\partial x^\gamma} + \frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^\gamma}\bigg|, \tag{2.6}$$

where $|$ corresponds to the substitution expressed in (2.3). It will be used in subsequent calculations. Hence, the equations of motion given by

$$\frac{\delta S_1}{\delta x^\gamma} = \frac{\partial \bar{L}(x^\alpha, \dot{x}^\alpha)}{\partial x^\gamma} - \frac{d}{dt}\left(\frac{\partial \bar{L}(x^\alpha, \dot{x}^\alpha)}{\partial \dot{x}^\gamma}\right) = 0 \tag{2.7}$$

provide, after rearranging the terms,

$$\left.\frac{\delta S_1}{\delta x^\gamma}\right| + \left.\frac{\delta S_1}{\delta x^1}\right|\frac{\partial f}{\partial x^\gamma} = 0. \tag{2.8}$$

The idea here is to show that one may insert the constraint $\Phi(x^i) = 0$ into the initial Lagrangian leading to an equivalent description. Let us consider the following action,

$$S_2 = \int dt \tilde{L}(x^i, \dot{x}^i, \lambda), \tag{2.9}$$

defined in an extended configuration space parametrized by x^i and λ, where

$$\tilde{L}(x^i, \dot{x}^i, \lambda) = L(x^i, \dot{x}^i) + \lambda \Phi(x^i). \tag{2.10}$$

The functions L and Φ are the same as the initial construction and λ is a Lagrange multiplier. Hence, the Euler-Lagrange equations are

$$\frac{\delta S_2}{\delta x^1} = 0 \Rightarrow \frac{\partial L(x^i, \dot{x}^i)}{\partial x^1} + \lambda\frac{\partial \Phi}{\partial x^1} = \frac{d}{dt}\left(\frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^1}\right), \tag{2.11}$$

$$\frac{\delta S_2}{\delta x^\gamma} = 0 \Rightarrow \frac{\partial L(x^i, \dot{x}^i)}{\partial x^\gamma} + \lambda\frac{\partial \Phi}{\partial x^\gamma} = \frac{d}{dt}\left(\frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^\gamma}\right), \tag{2.12}$$

$$\frac{\delta S_2}{\delta \lambda} = 0 \Rightarrow \Phi(x^i) = 0. \tag{2.13}$$

From (2.11), we find

$$\lambda = -\left(\frac{\partial \Phi}{\partial x^1}\right)^{-1}\left[\frac{\partial L(x^i, \dot{x}^i)}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^i}\right)\right]. \tag{2.14}$$

The substitution of (4.16) in (4.15) eliminates the λ-dependence of equations of motion,

$$\frac{\partial L(x^i, \dot{x}^i)}{\partial x^\gamma} - \frac{d}{dt}\left(\frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^\gamma}\right) - \left(\frac{\partial \Phi}{\partial x^1}\right)^{-1}\frac{\partial \Phi}{\partial x^\gamma}\left[\frac{\partial L(x^i, \dot{x}^i)}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L(x^i, \dot{x}^i)}{\partial \dot{x}^1}\right)\right] = 0 \tag{2.15}$$

Finally, from (2.13) and according to (2.2),

$$\Phi(x^i) = 0 \Leftrightarrow x^1 = f(x^\alpha). \tag{2.16}$$

Substitution of $x^1 = f(x^\alpha)$ into the constraint $\Phi(x^i) = 0$ gives the identity

$$\Phi(x^1 = f(x^\alpha), x^\alpha) \equiv 0, \tag{2.17}$$

whose derivative provides

$$0 = \frac{d}{dx^\gamma}\Phi(x^1 = f(x^\alpha), x^\alpha) = \frac{\partial \Phi(x^i)}{\partial x^1}\Big|\frac{\partial f}{\partial x^\gamma} + \frac{\partial \Phi(x^i)}{\partial x^\gamma}\Big|. \tag{2.18}$$

Then we have that

$$\frac{\partial f}{\partial x^\gamma} = -\left[\frac{\partial \Phi(x^i)}{\partial x^1}\Big|\right]^{-1}\frac{\partial \Phi(x^1)}{\partial x^\gamma}\Big|. \tag{2.19}$$

This expression appears in (2.15), which is now rewritten by eliminating x^1,

$$\left[\frac{\partial L(x^i,\dot{x}^i)}{\partial x^\gamma} - \frac{d}{dt}\left(\frac{\partial L(x^i,\dot{x}^i)}{\partial \dot{x}^\gamma}\right)\right]\Big| + \left[\frac{\partial L(x^i,\dot{x}^i)}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L(x^i,\dot{x}^i)}{\partial \dot{x}^1}\right)\right]\Big|\frac{\partial f}{\partial x^\gamma} = 0. \tag{2.20}$$

Since $\frac{d}{dt}(L|) = \frac{dL}{dt}|$, we arrive at

$$\frac{\delta S_1}{\delta x^\gamma}\Big| + \frac{\delta S_1}{\delta x^1}\Big|\frac{\partial f}{\partial x^\gamma} = 0. \tag{2.21}$$

These are the same equations of motion of the initial formulation, see (2.8). The equivalence between both constructions that have been developed so far becomes clearer if we compare the number of degrees of freedom in each description. The initial construction described by $\bar{L} = \bar{L}(x^\gamma, \dot{x}^\gamma)$ was formulated by eliminating x^1 with the previous knowledge of the constraint surface the model is immersed in. We are left $N-1$ degrees of freedom. On the other hand, the second one starts with $N+1$ variables. First, we have excluded λ from the description by using (2.11). Then, with the help of (2.13), x^1 was eliminated, see (2.16). These two steps left us with $N+1-2 = N-1$ degrees of freedom, as expected. This concludes the equivalence between S_1 and S_2. An application will be treated in the next subsection, when we consider the example of a particle over a 2-sphere.

2.2 A concrete example of constrained dynamics: particle over a 2-sphere

We will now discuss an application of the result found in the last Section. Actually, the main aim of these notes is the classical and NC descriptions of a free particle over a 2-sphere. Besides, the example of the particle over a 2-sphere will be used for a classical description of the Dirac spinning electron, see subsection 2.6.

Let m be the mass of the particle and $x^i = x^i(t)$, $i = 1,2,3$, its spatial coordinates. Since we want to formulate the particle evolution constrained to a 2-sphere, we take the following action,

$$S_\lambda(x^i) = \int_{t_1}^{t_2} dt\left[\frac{m}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \lambda(\delta_{ij}x^ix^j - a^2)\right], \tag{2.22}$$

where δ_{ij} stands for the delta Kronecker symbol and λ is again a Lagrange multiplier. S_λ has manifest $SO(3)$-invariance, which guarantees, for example, conservation of angular momentum. Equation of motion for λ gives the desired constraint

$$\delta_{ij}x^ix^j = a^2. \tag{2.23}$$

So, Eq. (2.22) in fact describes a free particle over a 2-sphere of radius a. On the other hand, we could exclude one of the variables with the help of (2.23),

$$x^3 = \pm\sqrt{a^2 - \delta_{\alpha\beta}x^\alpha x^\beta}, \tag{2.24}$$

where α, β run the values 1 and 2. Concerning the parametrization of the 2-sphere, we take the upper half plane $x^3 > 0$. Then, according to (2.3), we substitute (2.24) into the action for the free particle in a flat 3-dimensional space leading to

$$S_{ph} = \int dt \frac{m}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta, \tag{2.25}$$

where

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + \frac{x_\alpha x_\beta}{a^2 - \delta_{\alpha\beta}x^\alpha x^\beta}. \tag{2.26}$$

The action was named S_{ph} since we have eliminated the spurious degree of freedom x^3, obtaining an equivalent description of the particle over a 2-sphere in terms of physical variables x^1, x^2. It has a simple interpretation: since the particle is constrained to a 2-sphere, (2.25) describes a free particle in a Riemann space whose metric is given by $g_{\alpha\beta}$ [43]. The elimination of x^3 naturally led us to the concept of first fundamental form (or metric) [44]. In the limit $a \to +\infty$, we have a free particle in a flat bi-dimensional space. Namely, $g_{\alpha\beta} \to \delta_{\alpha\beta}$ and the Lagrangian originated from (2.25) becomes the kinetic energy of the particle,

$$\frac{m}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \to \frac{m}{2}\left[(\dot{x}^1)^2 + (\dot{x}^2)^2\right]. \tag{2.27}$$

We now turn our attention to the time evolution of the model. The dynamics is governed by the principle of least action. The minimization $\delta S_{ph} = 0$ gives the equation of motion

$$\ddot{x}^\alpha = G^\alpha{}_{\sigma\beta}\dot{x}^\sigma \dot{x}^\beta, \tag{2.28}$$

where

$$G^\alpha{}_{\sigma\beta} = g^{\alpha\gamma}\left(\frac{1}{2}\partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\gamma\beta}\right). \tag{2.29}$$

$g^{\alpha\gamma}$ corresponds to the inverse of the metric: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha{}_\gamma$ and $\partial_\gamma \equiv \frac{\partial}{\partial x^\gamma}$. Explicit calculation of G gives

$$G^\alpha{}_{\sigma\beta} = \frac{1}{2}\frac{x_\sigma \delta^\alpha{}_\beta - x_\beta \delta^\alpha{}_\sigma}{a^2 - \delta_{\gamma\rho}x^\gamma x^\rho} - \frac{x^\alpha g_{\sigma\beta}}{a^2}. \tag{2.30}$$

The first term of G is antisymmetric on $\sigma \leftrightarrow \beta$. Then it vanishes when contracted with the symmetric factor $\dot{x}^\sigma \dot{x}^\beta$ of (2.28). We are finally left with

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0, \tag{2.31}$$

and Γ is given by

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{x^{\alpha}}{a^2} g_{\beta\gamma}, \tag{2.32}$$

where (2.31) is the equation of a geodesic line: the particle chooses the trajectory with the shortest length. Moreover, the principle of least action gave us the Christoffel symbol or affine connection $\Gamma^{\alpha}{}_{\beta\gamma}$. Once again, the "static" concepts of differential geometry (geodesic line and second fundamental form Γ) were discovered via a dynamical realization. In the limit $a \to +\infty$, the equation of motion tends to

$$\ddot{x}^{\alpha} = 0, \tag{2.33}$$

which corresponds to the motion of a free particle (in flat bi-dimensional space) since $\Gamma^{\alpha}{}_{\beta\gamma} \to 0$, in accordance with our intuition.

In the next section we will solve the equations of motion (2.31). It will be accomplished in two different ways. The first one is by exploring the geometric setup that the model was constructed and the second one is by using the conserved currents obtained from Noether theorem [45].

2.3 Solution to equations of motion

Let us now obtain the solution of the equations of motion (2.31) in the commutative plane. It will be obtained via two different approaches. In the first one, we will use the geometric structure of the problem, i.e., since the particle is free, it is supposed to describe a circumference of radius a with constant angular velocity. Besides, we will also use the Noether theorem which provides two integrals of motion, which allow us to find the general solution of equations of motion.

2.3.1 Solving equations of motion: geometrical point of view

There is a standard way to solve the equations of motion in different models: if we know a particular solution, the general one is obtained by applying a group transformation in which the model is based on. For example, in [46], the author finds general spinors connected to an arbitrary state of motion of the Dirac electron by boosting plane wave solutions of the Dirac equation for a particle at rest. We will use the same prescription here. Initially, we take the following particular solution,

$$x^{i}(t) = \begin{pmatrix} 0 \\ a \sin \omega t \\ a \cos \omega t \end{pmatrix}, \tag{2.34}$$

that describes our free particle with constant (and arbitrary) angular velocity ω constrained to the 2-sphere of radius a. A direct calculation shows that it satisfies (2.31). We have restricted the motion to the plane x^2x^3. The general solution is achieved by three successive passive rotations around x^1, x^2 and x^3 axes. The rotations introduce three new and arbitrary parameters which, combined with ω, complete the necessary number of four constants of integration concerning the second order equation (2.31). Denoting $\mathscr{R}_{x^i}(\theta_j)$ the rotation around x^i-axis by an angle θ_j, we have

$$x^i(t) = \left[\mathscr{R}_{x^3}(\theta_3)\right]^i{}_j \left[\mathscr{R}_{x^2}(\theta_2)\right]^j{}_k \left[\mathscr{R}_{x^1}(\theta_1)\right]^k{}_l y^l(t), \tag{2.35}$$

where, for example,

$$\mathscr{R}_{x^1}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}. \tag{2.36}$$

The other matrices $\mathscr{R}_{x^2}(\theta_2)$ and $\mathscr{R}_{x^3}(\theta_3)$ are well-known from the $SO(3)$-group. The parameters θ_i are the Euler angles, taken in the $x^1x^2x^3$ convention. For different representations of the Euler angles, see [47, 48], for example.

So, for the general solution one can obtain that

$$x^i(t) = \begin{pmatrix} a\sin\theta_2\cos\theta_3\cos(\omega t+\theta_1) + a\sin\theta_3\sin(\omega t+\theta_1) \\ -a\sin\theta_2\sin\theta_3\cos(\omega t+\theta_1) + a\cos\theta_3\sin(\omega t+\theta_1) \\ a\cos\theta_2\cos(\omega t+\theta_1) \end{pmatrix}. \tag{2.37}$$

In Section 3, we have withdrawn the variable x^3 from the description. One may check that the expression above obeys the identity,

$$x^3(t) \equiv \sqrt{a^2 - (x^1(t))^2 - (x^2(t))^2}. \tag{2.38}$$

Then, the physical solution is given by the projection of $x^i = x^i(t)$ onto the plane x^1x^2. On this plane, the trajectory is an ellipse. In fact, with no loss of generality[1] we take to the solution

$$\tilde{x}^i(t) = \left[\mathscr{R}_{x^2}(\theta_2)\right]^i{}_k \left[\mathscr{R}_{x^1}(\theta_1)\right]^k{}_l y^l(t) \tag{2.39}$$

in the plane x^1x^2,

$$\tilde{x}^\alpha(t) = \begin{pmatrix} a\sin\theta_2\cos(\omega t+\theta_1) \\ a\sin(\omega t+\theta_1) \end{pmatrix}. \tag{2.40}$$

The trajectory is obtained by excluding the time of the parametric equations (2.40). It is given by

$$\frac{(\tilde{x}^1)^2}{a^2\sin^2\theta_2} + \frac{(\tilde{x}^2)^2}{a^2} = 1, \tag{2.41}$$

[1] The only effect of the last rotation $\mathscr{R}_{x^3}(\theta_3)$ is to make the semi-axes of the ellipse not coincident with the coordinate axes x^1 and x^2. Thus, for simplicity, we obtain the trajectory by looking to the solution $\tilde{x}^\alpha$ in (2.40).

which is the equation of an ellipse.

Finally, the general solution that we were looking for is given by the projection of (2.37) in the plane x^1x^2,

$$x^\alpha(t) = \begin{pmatrix} a\sin\theta_2\cos\theta_3\cos(\omega t+\theta_1) + a\sin\theta_3\sin(\omega t+\theta_1) \\ -a\sin\theta_2\sin\theta_3\cos(\omega t+\theta_1) + a\cos\theta_3\sin(\omega t+\theta_1) \end{pmatrix}, \tag{2.42}$$

whose trajectory is an ellipse. One then can ask about the possibility of interpreting this movement as generated by a central field. It will be discussed in section 2.4. Our next step consists of finding $x^\alpha = x^\alpha(t)$ with the help of conserved quantities.

2.3.2 Solving equations of motion: conserved quantities

One of the most impressive results in classical mechanics is the Noether theorem: if an action is invariant under a global transformation, then there is a related integral of motion, known as Noether charge. In our case, we may look at (2.22) or (2.25) since they are equivalent. Considering that (2.22) has global $SO(3)$-invariance,

$$x^i \to x'^i = R^i{}_j x^j;\ \text{where } R^T = R^{-1}. \tag{2.43}$$

It implies the conservation of angular momentum,

$$L_i = m\varepsilon_{ijk}x^j\dot{x}^k \Rightarrow \frac{dL_i}{dt} = 0. \tag{2.44}$$

One may also look at the expression (2.25), which is invariant under time translations

$$t \to t' = t + \tau. \tag{2.45}$$

In this case, the corresponding conserved quantity is

$$E = \frac{m}{2}g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta. \tag{2.46}$$

where E is considered as the energy of the particle. We now turn our attention to the equation of motion (2.31). It is immediately decoupled if we use (2.46),

$$\ddot{x}^\alpha + \frac{x^\alpha}{a^2}g_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0 \Rightarrow \ddot{x}^\alpha + \frac{2E}{ma^2}x^\alpha = 0. \tag{2.47}$$

Thus, the solution of (2.47) can promptly be written as

$$x^\alpha(t) = A^\alpha\sin(\Omega t + \varphi_\alpha);\ \Omega = \frac{\sqrt{2mE}}{ma} \tag{2.48}$$

where A^α and φ_α are arbitrary constants of integration. Substitution of the solution (2.48) into (2.44) and (2.46) gives, respectively,

$$\frac{L_3}{m\Omega} = -A^1A^2\sin(\varphi_2 - \varphi_1), \tag{2.49}$$

$$(A^1)^2 + (A^2)^2 = a^2 + \frac{L_3^2}{2mE}. \tag{2.50}$$

And (2.49) means that the angle between $x^1(t)$ and $x^2(t)$ is $\varphi_2 - \varphi_1$. If we assume that $\varphi_2 - \varphi_1 = \frac{\pi}{2}$, then the general solution may be achieved by rotating the particular solution with this restriction. So, first if we substitute (2.49) in (2.50) we have that

$$(A^1)^2 + (A^2)^2 = a^2 + \frac{(A^1)^2(A^2)^2}{a^2} \Rightarrow A^1 = a \Rightarrow A^2 = -\frac{L_3}{\sqrt{2mE}}. \tag{2.51}$$

We then have a particular solution $x_p^\alpha = x_p^\alpha(t)$, where x_p^1 and x_p^2 are perpendicular,

$$x_p^\alpha(t) = \begin{pmatrix} a\sin(\Omega t + \varphi_1) \\ -\frac{L_3}{\sqrt{2mE}}\cos(\Omega t + \varphi_1) \end{pmatrix}. \tag{2.52}$$

A final general solution can be obtained by rotating the particular solution above in an active way,

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \begin{pmatrix} x_p^1 \\ x_p^2 \end{pmatrix}, \tag{2.53}$$

that is,

$$x^\alpha(t) = \begin{pmatrix} a\cos\varphi_2\sin(\Omega t + \varphi_1) - \frac{L_3}{\sqrt{2mE}}\sin\varphi_2\cos(\Omega t + \varphi_1) \\ -a\sin\varphi_2\sin(\Omega t + \varphi_1) - \frac{L_3}{\sqrt{2mE}}\cos\varphi_2\cos(\Omega t + \varphi_1) \end{pmatrix}. \tag{2.54}$$

As expected, we have four constants of integration: $\varphi_{1,2}$, E and L_3. Equivalence between the two solutions (2.42) and (2.54) is manifest if we write

$$\begin{aligned} \omega &= \Omega, \\ \theta_1 &= \varphi_1, \\ \theta_3 &= \varphi_2 + \tfrac{\pi}{2}, \\ a\sin\theta_2 &= \tfrac{L_3}{\sqrt{2mE}}. \end{aligned} \tag{2.55}$$

In the next section, we will discuss a possible interpretation of the solution of the equations of motion in terms of an effective central potential induced by the space curvature.

2.4 Equivalence between a central force problem and the particle over a 2-sphere

The movement of the particle over the 2-sphere was completely described so far by the physical variables $x^\alpha(t)$, $\alpha = 1, 2$, see (2.42) or (2.54). Since the trajectory is an ellipse, one may think that it could be derived by a central field. So, the objective of this section is to show that the solution $x^\alpha(t)$ is equivalent to the one described by an isotropic harmonic oscillator. We already know the time evolution of the particle. The idea is, instead of solving a differential equation of motion, we would like to obtain it. For that, we will use polar coordinates $(x^1, x^2) \leftrightarrow (r, \theta)$

$$\begin{aligned} x^1 &= r\cos\theta \\ x^2 &= r\sin\theta \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} r &= \sqrt{(x^1)^2 + (x^2)^2} \\ \theta &= \arctan\left(\tfrac{x^2}{x^1}\right). \end{aligned} \tag{2.56}$$

For simplicity, we have used the solution (2.40). Let us construct the differential equations obeyed by the coordinates θ and r. We have that

$$\theta(t) = \arctan\left(\frac{x^2(t)}{x^1(t)}\right) = \arctan\left(\frac{\sin\Delta}{\sin\theta_2\cos\Delta}\right), \tag{2.57}$$

$$r(t) = a\sqrt{\sin^2\theta_2\cos^2\Delta + \sin^2\Delta}, \tag{2.58}$$

where we have used the shorthand notation $\Delta = \omega t + \theta_1$. First time derivative of (2.57) gives

$$\dot{\theta}(t) = \frac{\omega a^2 \sin\theta_2}{r^2(t)} = \frac{L_3}{mr^2(t)}, \tag{2.59}$$

since the angular momentum L_3 is given by

$$L_3 = m(\dot{x}^2 x^1 - \dot{x}^1 x^2) = m\omega a^2 \sin\theta_2. \tag{2.60}$$

We turn our attention to the radial variable. It is a tedious but rather direct calculation to obtain the second order time derivative of Eq. (2.58). We have

$$\dot{r}(t) = \frac{\omega a^2 \sin 2\Delta(1-\sin^2\theta_2)}{2r}. \tag{2.61}$$

The second time derivative reads

$$\ddot{r} = \frac{\omega^2 a^2 \cos 2\Delta(1-\sin^2\theta_2)}{r} - \frac{\omega a^2 \sin 2\Delta(1-\sin\theta_2)}{2r^2}\dot{r}. \tag{2.62}$$

Substituting $\dot{r}(t)$ into the expression above, one finds after rearranging the terms,

$$\ddot{r} = \frac{\omega^2 a^4}{r^3}[-(\cos^2\Delta\sin^2\theta_2 + \sin^2\Delta)^2 + \sin^2\theta_2(\sin^2\Delta + \cos^2\Delta)^2], \tag{2.63}$$

which multiplied by the mass of the particle becomes

$$m\ddot{r} = -\omega^2 r + \frac{L_3^2}{m^2 r^3} \Rightarrow m\ddot{r} = -m\omega^2 r + \frac{L_3^2}{mr^3}. \tag{2.64}$$

Eqs. (2.59) and (2.64) are exactly the ones obeyed by a particle in a central field [47]. Eq. (2.64) may be seen as the second Newton's law for a particle in a isotropic harmonic oscillator. The term $\frac{L_3^2}{mr^3}$ corresponds to the centrifugal force always present when one writes a central force in polar coordinates. The first term, that has been associated with the harmonic oscillator, may be considered as an effective force due to the curved space the particle is constrained to. In fact, we construct the scalar or total curvature of the surface

$$R = g^{\alpha\beta}(\partial_\gamma {\Gamma^\gamma}_{\alpha\beta} - \partial_\beta {\Gamma^\gamma}_{\alpha\gamma} + {\Gamma^\gamma}_{\alpha\beta}{\Gamma^\lambda}_{\lambda\gamma} - {\Gamma^\gamma}_{\alpha\lambda}{\Gamma^\lambda}_{\beta\gamma}). \tag{2.65}$$

Using the Christoffel symbols (2.32) and the inverse of the metric

$$g^{\alpha\beta} = \delta^{\alpha\beta} + \frac{x^\alpha x^\beta}{a^2}, \tag{2.66}$$

one obtains

$$R = \frac{2}{a^2}. \tag{2.67}$$

It turns out that the constant force of second Newton's law (2.64) is proportional to the total curvature,

$$k = m\omega^2 = m\frac{2mE}{m^2a^2} = RE. \tag{2.68}$$

Thus the movement of the free particle over a 2-sphere projected in x^1x^2-plane is equivalent to the movement described by a particle in a central effective potential

$$V_{eff}(r) = \frac{RE}{2}r^2 + \frac{L_3^2}{2mr^2}, \tag{2.69}$$

as stated and both potentials, $V(r) \sim \frac{1}{r}$ and $V(r) \sim r^2$ produce the same trajectory, i.e., an ellipse.

2.5 Hamiltonization of constrained systems: interpretation of the Dirac brackets based on geometric grounds

Since our discussion on the dynamics of a constrained system has been restricted to the Lagrangian formalism, the objective of this section is based on the hamiltonization of the Lagrangian L_λ. At the time when Dirac proposed his formalism, it was not completely understood how to introduce constraints into the Hamiltonian formalism [23], which is a solved problem in current days [43, 49, 50, 51]. Hamiltonization of L_λ leads to the so-called Dirac brackets and we will provide its geometric interpretation. The construction of the Hamiltonian concerning (2.22) begins with the definition of the conjugate momenta

$$p_A \equiv \frac{\partial L}{\partial \dot{q}^A}, \tag{2.70}$$

where we wrote collectively $q^A = (x^i, \lambda)$. According to the formalism, we can use the expression of conjugate momenta to obtain the maximum number of velocities as functions of momenta and configuration variables,

$$p_A = \frac{\partial L}{\partial \dot{q}^A} \Leftrightarrow \begin{cases} p_i = \frac{\partial L}{\partial \dot{x}^i} \Leftrightarrow \dot{x}^i = \frac{1}{m}p_i, \\ p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} \Rightarrow p_\lambda = 0. \end{cases} \tag{2.71}$$

Let us define $T_1 \equiv p_\lambda = 0$ and call it a primary constraint. The complete Hamiltonian is defined in extended phase space q^A, p_A, v

$$\begin{aligned} H &= p_A\dot{q}^A - L + vp_\lambda \\ &= \frac{1}{2m}p_i^2 - \lambda[(x^i)^2 - a^2] + vp_\lambda, \end{aligned} \tag{2.72}$$

where v is a Lagrange multiplier and all velocities enter into H according to (2.71). Let us write the equations of motion via Poisson brackets again such that

$$\dot{q}^A = \{q^A, H\} \Rightarrow \begin{cases} \dot{x}^i = \frac{1}{m} p_i, \\ \dot{\lambda} = v, \end{cases} \tag{2.73}$$

$$\dot{p}_i = \{p_i, H\} = 2\lambda x^i. \tag{2.74}$$

Since a constraint must be constant, one obtains the following chain of secondary constraints

$$T_2 = \dot{p}_\lambda = \{p_\lambda, H\} \Rightarrow T_2 = (x^i)^2 - a^2 = 0, \tag{2.75}$$
$$T_3 = \dot{T}_2 = \{T_2, H\} \Rightarrow T_3 = x^i p_i = 0, \tag{2.76}$$
$$T_4 = \dot{T}_3 = \{T_3, H\} \Rightarrow T_4 = \frac{1}{m} p_i^2 + 2\lambda (x^i)^2. \tag{2.77}$$

Finally, the evolution in time of T_4 allows us to find the Lagrange multiplier v,

$$v = 0. \tag{2.78}$$

The matrix $T_{ab} = \{T_a, T_b\}$; $a, b = 1, 2, 3, 4$ is invertible, then according to the Dirac terminology, the constraints are called second class (actually, the existence of T_{ab}^{-1} is the reason why all multipliers have been found [43]). The Dirac brackets are

$$\{A, B\}^* = \{A, B\} - \{A, T_a\} T_{ab}^{-1} \{T_b, B\}. \tag{2.79}$$

So, the equations of motion are defined over the constraint surface and one may forget about the equations $T_a = 0$. They read,

$$\dot{Y} = \{Y, H_0\}^*, \tag{2.80}$$

where $Y = (x^i, p_i)$ since the sector (λ, p_λ) may be omitted and $H_0 \equiv H - v p_\lambda$. The basic Dirac brackets for the (x^i, p_i)-sector have the form

$$\{x^i, x^j\}^* = 0, \tag{2.81}$$
$$\{x^i, p^j\}^* = \delta^{ij} - \frac{x^i x^j}{a^2}, \tag{2.82}$$
$$\{p^i, p^j\}^* = -\frac{1}{a^2}(x^i p^j - x^j p^i). \tag{2.83}$$

Since the equations of motion described via Lagrangian formalism give the proper time evolution of the particle over the surface as well as the Lagrangian and Hamiltonian formulations being equivalent [49], one expects a relationship between Christoffel symbols and the Dirac bracket. To see this, first we decouple the equation for x^i,

$$m\dot{x}^i = p^i \Rightarrow m\ddot{x}^i = 2\lambda x^i. \tag{2.84}$$

With the help of the constraints T_2, T_4 and (2.73), we obtain that

$$\lambda = -\frac{m(\dot{x}^i)^2}{2a^2}. \tag{2.85}$$

Then,

$$\ddot{x}^i = -\frac{(\dot{x}^i)^2}{a^2}x^i. \tag{2.86}$$

On the other hand, we may write

$$\ddot{x}^i = \frac{1}{m}\{p^i, H_0\}^*|, \tag{2.87}$$

where $|$ denotes substitution of p_j in terms of position and velocity variables, see (2.71). The α-sector ($\alpha = 1, 2$) of equation (2.86) coincides with equations of motion (2.31) of the Lagrangian formalism. Comparing it with (2.87), one finds

$$\{H_0, p^\alpha\}^*| = m\Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = -\ddot{x}^\alpha. \tag{2.88}$$

This calculation that compares equations of motion in both Lagrangian and Hamiltonian formalisms shows the intrinsic relation between Christoffel symbols and Dirac brackets, as these structures are the ones responsible for the time evolution of the particle in each formalism.

2.6 Application: spinning particle

The complete understanding of electron spin was accomplished in the realm of quantum electrodynamics. If we consider the Dirac equation

$$i\hbar\partial_t\Psi = \hat{H}\Psi; \ \hat{H} = c\alpha^i\hat{p}_i + mc^2\beta, \tag{2.89}$$

as the one-particle equation in Relativistic Quantum Mechanics then, in the Heisenberg picture, the position operators experience a quivering motion [52]

$$x^i = a^i + bp^it + c^i\exp\{-\frac{2iH}{\hbar}t\} \tag{2.90}$$

that may be considered a superposition of a rectilinear movement with an harmonic one, with high frequency $\frac{2H}{\hbar} \sim \frac{2mc^2}{\hbar}$. This harmonic oscillation was named *Zitterbewegung* by Schrödinger [24]. In recent literature, a model has been proposed with commuting variables that produces the Dirac equation through quantization [53]. Analysis of the classical counterpart of the model leads to the so-called *Zitterbewegung*, also experienced by spin variables. In order to provide spacetime interpretation for the evolution of the classical position and spin coordinates, they were combined to produce configuration coordinates whose dynamics is given by (see details in [54]),

$$\tilde{x}^i(t) = x^i + c\frac{p^i}{p^0}t, \tag{2.91}$$

$$J^i(t) = \frac{1}{2|p|}(A^i \cos\omega t - B^i \sin\omega t), \tag{2.92}$$

with A^i, B^i, p^μ being some constants, $|p| \equiv \sqrt{-p_\mu p^\mu}$ and ω has the same order of magnitude as the Compton frequency. They evolve similarly to the center-of-mass and relative position of two-body problem in a central field. The potential turns out to be $V(J) \sim J^2$; $J = |J^i|$. Assuming that (2.91) and (2.92) are the position variables for the electron, then J^i describes an ellipse with restricted size (a particular feature of the model restricts the magnitude of A^i and B^i as well as their direction, since $p_i A^i = p_i B^i = 0$, the center-of-mass moves perpendicularly to the plane of oscillations). According to the previous sections, we interpret J^i as the physical variables for the motion over a 2-sphere. This may explain the physical origin of the *Zitterbewegung* if we assume that the electron has an internal structure [55]. It seems that Dirac himself believed that the electron was not an elementary particle, see [56].

The idea of a composed electron goes back to the seminal paper by Dirac on the unitary irreducible particle representations of the Anti-de Sitter group [57]. Actually, in this work, he found two remarkable representations of SO(2,3), the isometry group of Anti-de Sitter space AdS_4. Those representations do not have a counterpart in Poincaré group; they are unique to SO(2,3). This means that, whenever the (Riemann) curvature of AdS_4 goes to zero, these two representations may be combined in order to construct one of the unitary irreducible representations of Poincaré group in terms of one-particle states. He called these representations singletons. Currently, singleton physics is an active research area [58]. Moreover, preons appear as “point-like"particles are perceived as being subcomponents of quarks and leptons. This term was coined by Jogesh Pati and Abdus Salam in their 1974 paper [59]. Preon models set out as an attempt to describe particle physics in a more fundamental level than the Standard Model [60]. In these preonic models, one postulates a set of fewer fundamental particles than those of the Standard Model, together with the interactions governing the dynamics of these fundamental particles. Based on these laws, preon models try to explain some physics beyond the Standard Model, often producing new particles and a number of phenomena which do not belong to the Standard Model.

2.7 Noncommutative classical mechanics in a curved phase-space

As we saw previously, the canonical NCy is described by the following algebra

$$[\hat{x}^i, \hat{x}^j] = i\,\theta^{ij} \qquad ; \qquad [\hat{x}^i, \hat{p}_j] = i\,\delta^i_j \qquad ; \qquad [\hat{p}_i, \hat{p}_j] = 0\ , \tag{2.93}$$

where we are using that $\hbar = 1$ and θ^{ij}'s are c-numbers with the dimensionality of (length)^2. Let us assume that this so-called NC parameter is within the Planck's area order, i.e.,

$l_P^2 = \hbar G/c^3$, so we have that the tensor θ^{ij} must be of G/c^3 order. Hence, in the classical limit, the symplectic framework will not have $\hbar$ [25]. This result agrees with this kind of limit. At the classical level, the quantum mechanical commutator is substituted by the Poisson bracket via

$$\left[\hat{A},\, \hat{B}\right] \longrightarrow i\,\{A,\, B\} \tag{2.94}$$

and consequently, the classical limit of (2.93) is

$$\{x^i,\, x^j\} \,=\, \theta^{ij} \qquad ; \qquad \{x^i,\, p_j\} \,=\, \delta^i_j \qquad ; \qquad \{p_i,\, p_j\} \,=\, 0 \tag{2.95}$$

where θij is an antisymmetric constant matrix and the Poisson bracket must have the same properties as the quantum mechanical commutator (bilinear, antisymmetric, Leibniz rules, Jacobi identity). In this section we will assume a symplectic structure given by (2.95) in order to obtain the corresponding equations of motion. It is important to say that there are NC formulations where the momenta commutator (Poisson bracket) is not zero. But we will not analyze it here.

We will assume a symplectic structure for the classical mechanics of a particle in a curved phase-space. The target geometry is the 2-sphere described above. We will demonstrate that there is a correction term added to Newton's second law thanks to the curved configuration of the phase-space, which shows that the space configuration alone can bring consequences to the result. On the other hand, we will see that in a flat space, what causes a NC correction is the potential function, which is a standard result in NC classical mechanics. In the 2-sphere curved space we will see that there is a NC correction without the existence of a potential effect over the particle. This result is coherent with the one obtained here that established ana analogy between the curvature of a 2-sphere and a central field.

Let us begin by describing the origin of the NC contribution in the generalized (without a specific potential) Newton's second law [25, 61, 62]. We can define a theory as being formulated by a set of canonical variables ξ^a, where $a \,=\, 1, \ldots, 2n$ combined with a symplectic structure $\{\xi^a, \xi^b\}$. This structure can be extended in order to accommodate arbitrary function of ξ^a such as

$$\{F, G\} \,=\, \{\xi^a, \xi^b\} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b} \tag{2.96}$$

where F and G are two arbitrary function of phase-space and the repeated indices are summed from now on. Eq.(2.96) can be used, of course, in classical mechanical systems [25, 61, 62] as the one we will analyze in this thesis.

In Hamiltonian systems, we can use the structure given in (2.96) to write the equations of motion for a Hamiltonian given by $H = H(\xi^a)$ such that

$$\dot{\xi}^a = \{\xi^a, H\} \tag{2.97}$$

and for a generalized function F defined in this space we can write that

$$\dot{F} = \{F, H\}. \tag{2.98}$$

In our case, we will consider a phase-space given by the physical variables x and y and so, $\xi = (x, p_x, y, p_y)$. The algebra between these coordinates is

$$\{x, y\} \;=\; \theta, \qquad \{x, p_x\} \;=\; \{y, p_y\} \;=1 \qquad \{p_x, p_y\} \;=\; 0, \tag{2.99}$$

Let us consider two arbitrary functions F and G, defined on the phase-space. Using Eqs. (2.96) and (2.99) we have that

$$\{F, G\} \;=\; \theta^{ij}\,\frac{\partial F}{\partial x^i}\frac{\partial G}{x^j} \;+\; \frac{\partial F}{\partial x^i}\frac{\partial G}{p_i} \;-\; \frac{\partial F}{\partial p_i}\frac{\partial G}{x^i} \tag{2.100}$$

where $i, j \;=\; x, y$. For example, if we have a Hamiltonian of the standard form with $\xi \;=\; (x^i, p_i)$ such that

$$H \;=\; \frac{p_i p^i}{2m} \;+\; V(x) \tag{2.101}$$

using (2.98) and (2.100) we have the equations of motion given by

$$\dot{x}^i \;=\; \{x^i, H\} = \theta\frac{\partial H}{\partial x_i} \;+\; \frac{\partial H}{\partial p_i} \Longrightarrow \dot{x}^i \;=\; \frac{p^i}{m} \;+\; \theta^{ij}\frac{\partial V}{\partial x^j}$$

and analogously

$$\dot{p}_i \;=\; -\frac{\partial V}{\partial x^i}\;. \tag{2.102}$$

Notice from both these equations that an obvious conclusion is that if $V = 0$ (free particle) we have $p_i =$ constant and x^i is a linear function of time. Hence, the second term of $\dot{x}^i$ is connected to V, an external field. We can understand that the dynamics of the framework is ruled by the the perturbation caused by this external field in the NC phase-space. Newton's second law can be obtained analogously (from Eq. (2.98)) and the result is

$$m\,\ddot{x}^i \;=\; -\frac{\partial V}{\partial x_i} \;+\; m\theta^{ij}\,\frac{\partial^2 V}{\partial x^j \partial x_k}\dot{x}_k\;. \tag{2.103}$$

This result was used to investigate several models in physics [63]. Here, we want to verify how the phase-space curvature affects the NC contribution. We can see that this new force can be understood, analogously to (2.102), as the result of a perturbation in the classical phase-space as a consequence of an external field.

In our case, we want to discuss the NC approach for the free particle in a flat 3D space which has the Lagrangian given by

$$L_{ph} = \frac{m}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{2.104}$$

where $g_{\alpha\beta}$ is given in (2.26). From (2.104) we have that

$$\begin{aligned} p_x &= m\dot{x} + \frac{mx(x\dot{x} + y\dot{y})}{a^2 - x^2 - y^2} \\ p_y &= m\dot{y} + \frac{my(x\dot{x} + y\dot{y})}{a^2 - x^2 - y^2} \end{aligned} \tag{2.105}$$

where we have used that $x_1 = x$ and $x_2 = y$. From Eqs. (2.104) and (2.105), the Hamiltonian is given by

$$H = \frac{1}{2ma^2}\Big[(a^2 - x^2)p_x^2 + (a^2 - y^2)p_y^2 - 2p_x p_y xy\Big] \tag{2.106}$$

and our set of symplectic variables is given by $\xi = (x, y, p_x, p_y)$, as we said before. Using Eqs. (2.98)-(2.100) and the Hamiltonian in (2.106) we have the NC equations of motion

$$\begin{aligned} \dot{x} &= \frac{1}{ma^2}\Big[(a^2 - x^2)p_x - xyp_y - \theta\Big(yp_y^2 + xp_x p_y\Big)\Big] \\ \dot{y} &= \frac{1}{ma^2}\Big[(a^2 - y^2)p_x - xyp_x - \theta\Big(xp_x^2 + yp_x p_y\Big)\Big] \\ \dot{p_x} &= \frac{1}{ma^2}\Big(xp_x^2 + p_x p_y y\Big) \\ \dot{p_y} &= \frac{1}{ma^2}\Big(yp_y^2 + p_x p_y x\Big) \ . \end{aligned} \tag{2.107}$$

Notice that when $\theta = 0$ we have the standard commutative phase-space equations of motion. Secondly, from (2.107) we can see the effect of a curved phase-space. For a free particle we must have $\dot{p}_x = \dot{p}_y = 0$, and this is the result of a free particle in a flat phase-space. However, before the calculation of $\dot{p}_x$ or $\dot{p}_y$ we can see the curvature effect already in $\dot{x}$ and $\dot{y}$. In other words, we do not need the values of $\dot{p}_x$ and $\dot{p}_y$ to know that the curvature plays a kind of potential in order to perturb the NC calculations [27].

It is important to say that if we have NCy in the momentum bracket of Eq. (2.95) and (2.98) we would have a θ-term in the momentum dynamics of (2.107).

After long algebra the NC Newton's second law for our particle on the 2-sphere is

$$m\ddot{x} = -\frac{1}{ma^4}\Big[x(a^2 - x^2)p_x^2 - 2x^2yp_xp_y - x(a^2 + y^2)p_y^2\Big] - \frac{\theta}{ma^2}(p_x^2 + p_y^2)p_y \tag{2.108}$$

and

$$m\ddot{y} = -\frac{1}{ma^4}\Big[y(a^2 - y^2)p_y^2 - 2xy^2p_xp_y - y(a^2 + x^2)p_x^2\Big] - \frac{\theta}{ma^2}(p_x^2 + p_y^2)p_x \tag{2.109}$$

and curiously we saw that in (2.103) the NC correction depends on the background space through the θ^{ij} parameter and also on the variations of the potential. This result could lead us to think that for our free particle, the NC corrections would be zero, as the expression obtained in [25] (Eq. (2.103)) could also indicate this). However, we can see in (2.108)-(2.109) that the curvature of the space originates a NC correction as well, in spite of a zero potential. In other words, we understand Eqs. (2.108) and (2.109) as a new NC Newton's second law. At the final terms of Eqs. (2.108) and (2.109) we can realize the correction due to the NC rule. This correction term relies on the background space through the NC θ-parameter. However, we can see the 2-sphere term represented by a, which is an expected result.

3 Canonical noncommutativity and quantum field theory

3.1 Canonical NCy and quantum field theory

3.1.1 NC quantum field theory

The theoretical framework for studying QFT in the NC spaces is called NC field theory (NCFT) and it may be a relevant physical model at scales between l_P ($\simeq 1.6 \times 10^{-33}cm$) and l_{LHC} ($\simeq 2 \times 10^{-18}cm$). In fact, one of the main threads of research in this field has been related to studies of energetic cosmic rays, as we will discuss further below. In the following we will study this relationship in some detail. These field theories provide fruitful avenues of investigation for several reasons, that will be explained in more depth below.

Firstly, some QFT's are better behaved on NC spacetime than on ordinary spacetime. In fact, some are completely finite, even non-perturbatively. In this manner spacetime NCy presents an alternative fomalism to supersymmetry or string theories in some sense. Secondly, it is a useful arena for studying physics beyond the standard model, and also for standard physics in strong external fields. Thirdly, it sheds light on alternative lines of attack to address various fundamental issues in QFT, for instance the renormalization and axiomatic programs. Finally, it naturally relates field theory to gravity. Since the field theory may be easier to quantize, this may provide significant insights into the problem of quantizing gravity.

Nowadays, in accordance with the Hopf-algebraic classification of all deformations of relativistic and non-relativistic symmetries, one can distinguish three kinds of spacetime NCy [64, 65]

1- Canonical (soft) deformation

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = i\theta^{\mu\nu}, \tag{3.1}$$

with tensor $\theta^{\mu\nu}$ being constant and antisymmetric ($\theta^{\mu\nu} = -\theta^{\nu\mu}$).

2- Lie-algebraic case

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = i\theta^{\mu\nu}_{\rho}\hat{x}^{\rho}, \tag{3.2}$$

with particularly chosen constant coefficients $\theta^{\mu\nu}_{\rho}$. This kind of spacetime modification is represented by κ-Poincaré and κ-Galilei Hopf algebras.

3- Quadratic deformation

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right] = i\theta^{\mu\nu}_{\rho\sigma}\hat{x}^{\rho}\hat{x}^{\sigma}, \tag{3.3}$$

with constant coefficients $\theta^{\mu\nu}_{\rho\sigma}$. The **hat** symbol " ˆ " above the variables indicates that they are NC variables.

According to this classification, the NCy that has emerged within the string theory belongs

to the canonical NCy. One of the main consequences of canonical NCy is absence of full Lorentz invariance. Both θ^{i0} and $\epsilon^{ijk}\theta_{ij}$ are fixed three-vectors that define preferred directions in a given Lorentz frame. The NC QFTs based on this category possess symmetry under various twisted Poincaré algebras, depending on the structure of θ [66, 67]. The advantage of using the twisted Poincaré language for constructing physical theories is that, in spite of the lack of full Lorentz symmetry, the fields carry representations of the full Lorentz group [68] and the spin-statistics relation is still valid; the deformation then appears in the product of the fields (interaction terms).
As there is no empirical data to support the violation of Lorentz symmetry in nature, to loose the Lorentz invariance property is not a good thing for any theory, in the opinion of some scientists. For this reason Doplicher, Fredenhagen and Roberts (DFR), have suggested that the NC parameter may not be a constant one and in this way the Lorentz invariance would be recovered [69]. We will see that the DFR algebra has been proposed based on issues that come from general relativity and quantum mechanics. The authors claim that very accurate measurement of spacetime position of a test particle could transfer such amount of energy to it that at least theoretically could be sufficient to create a gravitational field that, *a priori* could trap photons. Analyzing the limitations of this position measurement using a semi-classical approximation leads to uncertainty relations among spacetime coordinates

$$\Delta x^0 \sum_{i=1}^{3} \Delta x^i \gtrsim l_P^2 \quad ; \qquad \sum_{1 \geqslant j \geqslant k \geqslant 3} \Delta x^j \Delta x^k \gtrsim l_P^2. \tag{3.4}$$

These relations can be traced back to the commutation relations among coordinates (though not uniquely)

$$\left[\hat{x}^\mu, \hat{x}^\nu\right] = iQ^{\mu\nu}, \tag{3.5}$$

where Q is a tensor whose components $Q^{\mu\nu}$ commute with all coordinates. Thus, the presence of classical gravitation makes the spacetime effectively NC and this feature should be present in any quantum theory of gravitation.

The results appearing in [70] are explored by some authors [71, 72, 73, 74, 75]. Some of them prefer to start from the beginning by adopting DFR algebra, which essentially assumes (3.5) as well as the vanishing of the triple commutator among the coordinate operators. As it was cleared above, the DFR algebra is based on the principles coming from general relativity and quantum mechanics. In addition to (3.5) it also assumes that

$$[\hat{x}^\mu, Q^{\alpha\beta}] = 0. \tag{3.6}$$

An important point in DFR algebra is that the Weyl representation of NC operators obeying (3.5) and (3.6) keeps the usual form of the Moyal-Weyl product, and consequently the form of the usual NCFT's, although the fields have to be considered as depending not only on x^μ but also on $\theta^{\alpha\beta}$. The argument is that very accurate measurements of spacetime localization could transfer to test particles energies sufficient to create a gravitational field that in principle could trap photons. This possibility is related with spacetime uncertainty relations that can be derived from (3.5) and (3.6) as well as from the quantum conditions

$$\theta_{\mu\nu}\theta^{\mu\nu} = 0, \qquad \left(\frac{1}{4}\,{}^{*}\theta^{\mu\nu}\theta_{\mu\nu}\right)^2 = \lambda_P^8, \tag{3.7}$$

where ${}^{*}\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$ and λ_P is the Planck length.

These operators are seen as acting on a Hilbert space $\mathscr{H}$ and this theory implies in extra compact dimensions [69]. The use of conditions (3.7) in [70, 72, 73, 74, 75] would bring trivial consequences, since in those works the relevant results strongly depend on the value of θ^2, which is taken as a mean with some weight function $W(\theta)$. They use in this process the Seiberg–Witten [16] transformations that is explained previously. Of course those authors do not use (3.7), since their motivations are not related to quantum gravity but basically with the construction of a NCFT which keeps Lorentz invariance.

A nice framework to study different aspects of NCy is given by the so called NC quantum mechanics (NCQM), due to its simpler approach. There are many interesting works in NCQM but in most of these works, the object of NCy θ^{ij} (where $i, j = 1, 2, 3$), which essentially is the result of the commutation of two coordinate operators, is considered as a constant matrix.

In NCQM, although time is a commutative parameter, the space coordinates do not commute. However, the objects of NCy, θ, are not considered as Hilbert space operators. As a consequence the corresponding conjugate momenta is not introduced, because, as well known, it is important to implement rotation as a dynamical symmetry [76]. As a result, the theories are not invariant under rotations.

In [77], the author promoted an extension of the DFR algebra to a non-relativistic QM in a trivial way, but keeping consistency. The objects of NCy were considered as true operators and their conjugate momenta were introduced. This permits us to display a complete and consistent algebra among the Hilbert space operators and to construct generalized angular momentum operators, obeying the $SO(D)$ algebra, and in a dynamical way, acting properly in all the sectors of the Hilbert space.

In a recent work [78], the authors have indicated that in fact if the NC parameter is a coordinate of this new Hilbert space, an associate momentum is directly connected to it. Namely the new phase space would be formed by the original coordinates and the (θ, π) new pair.

3.2 Noncommutative gauge theory

Gauge theories are crucially important when building a realistic physical model and are the main ingredients of standard model of particle physics. So, in order to obtain any real results out of the NC field theory, the notion of gauge symmetry had to be generalized to the NC setting. Since gauge symmetries are essentially local, generalizing them to the nonlocal NC spacetime is highly nontrivial.

There are two methods to construct gauge field theories in NC spacetime. First uses the Seiberg-Witten map, obtained from string theory [16], which maps a NC gauge theory to a commutative gauge theory. In the second, one uses a NC generalization of a gauge group and the $\star$-product to construct a gauge theory in the framework of NC field theory. Both methods have been further developed and they offer some flexibility in their approaches. In this chapter we shall study just the Seiberg-Witten method briefly in the case of the constant θ and then we will construct a NC version of a Non-Yang-Mills gauge theory with $SU(N)$ gauge group. The reader with an interest in field theoretical approach can refer to [79, 80, 66, 67, 68].

Until now we have studied Lorentz-invariant NC spacetime which the parameter of NCy was an operator valued object but now we will take a look at the cases where the NC spacetime is considered to be the canonical one, i.e. the parameter of NCy be a real valued *constant* matrix. In this type of NCy the Lorentz invariance is violated.

For future use the Moyal $\star$-product and the Moyal bracket [2]are naturally generalized for the algebra of matrix-valued functions $M_{n\times n}\otimes A_\theta$, i. e., for two arbitrary functions $f(x)$ and $g(y)$ we have

$$(f(x)\star g(y))_{ij} = f(x)_{ik}\star g(y)_{kj}. \tag{3.8}$$

The Hermitian conjugation for the algebra $M_{n\times n}\otimes A_\theta$ can be defined by the usual Hermitian conjugation of matrices $\left(f(x)^\dagger\right)_{ij} = \left(f(x)^\star_{ji}\right)$ and by the definition that the $\star$-product behaves under the operation

$$(f(x)\star g(x))^\dagger = g(x)^\dagger \star f(x)^\dagger. \tag{3.9}$$

3.2.1 The Seiberg-Witten map and universal enveloping algebra

After a quantization process, the open string theory in a constant antisymmetric background field, with string end points constrained on D-branes, by using the Pauli-Villars and the point-splitting regularization, one obtains a commutative or NC gauge theory, respectively. The Seiberg-Witten (SW) map provides a correspondence between these two gauge theories, which should be equivalent, since a well-defined quantum theory does not depend on the regularization technique.

The SW map, as originally proposed, is a map between the NC $U_\star(N)$ gauge theory, described by $\hat{A}$ and Λ as gauge field and gauge transformations, respectively and the

[2] For a review of Moyal-Weyl product refer to A

corresponding ordinary commutative $u(N)$-matrix valued functions A and Λ. In this approach it is argued that, because most of the gauge theories on NC spaces cannot be constructed with Lie algebra valued infinitesimal gauge transformations, the infinitesimal gauge transformations should instead, be taken to be enveloping algebra valued. The idea is to bypass the difficulties in constructing NC gauge groups by letting the generators of the gauge transformations and the gauge fields to take values in the universal enveloping of the corresponding gauge algebra. The main problem with this approach is that enveloping algebras are infinite dimensional, which means that simply the numbers of both gauge transformation parameters and the gauge fields are infinite.

The gauge transformation parameters and the gauge fields can, however, be defined to be functions of the corresponding Lie algebra valued objects — the functions being obtained through the SW maps —, so that their numbers are the same as in the corresponding commutative gauge theories.

Let us consider the NC version of a gauge theory of a generic non-Abelian gauge algebra, say the algebra $su(n)$, with the matter fields $\hat{\psi}$ and the gauge fields $\hat{A}_\mu$. The infinitesimal local gauge transformations are

$$\hat{\delta}_{\hat{\Lambda}}\hat{\psi} = i\rho_\psi(\hat{\Lambda}(x)) \star \hat{\psi} \tag{3.10}$$
$$\hat{\delta}_{\hat{\Lambda}}\hat{A}_\mu = \hat{\partial}_\mu\hat{\Lambda}(x) + i\left[\hat{\Lambda}(x), \hat{A}_\mu\right]_\star \tag{3.11}$$

where the NC infinitesimal gauge transformation parameter $\hat{\Lambda}$ is valued in a universal enveloping of the gauge algebra $\mathcal{U}(su(n))$ and ρ_ψ is the matter representation of $\mathcal{U}(su(n))$[3]. It should be noted that there is no gauge symmetry group, since this gauge symmetry is only defined for infinitesimal gauge transformations[4]. Generally speaking, the gauge transformation parameter $\hat{\Lambda}$ cannot be Lie algebra valued, because the commutator of two Lie algebra valued parameters $\hat{\Lambda} = \hat{\Lambda}_i T_i$ and $\hat{\Sigma} = \hat{\Sigma}_i T_i$ does not close in the Lie algebra with the gauge transformations

$$\left[\hat{\Lambda}\,,\,\hat{\Sigma}\right]_\star = \frac{1}{2}\{\hat{\Lambda}_i\,,\,\hat{\Sigma}_j\}_\star \underbrace{[T_i\,,\,T_j]}_{if_{ijk}T_k} + \frac{1}{2}\underbrace{\left[\hat{\Lambda}_i\,,\,\hat{\Sigma}_j\right]_\star}_{\neq 0}\{T_i\,,\,T_j\}. \tag{3.12}$$

Therefore, we have to use fields and gauge transformations that are $\mathcal{U}(su(n))$-valued. The gauge fields $\hat{A}_\mu$ have to be in the adjoint representation of $\mathcal{U}(su(n))$. The gauge covariant derivative and the field strength are given by

$$\hat{D}_\mu\hat{\psi} = \partial_\mu\hat{\psi} - i\rho_\psi(\hat{A}_\mu) \star \hat{\psi} \tag{3.13}$$
$$\hat{F}_{\mu\nu} = \partial_{[\mu}\hat{A}_{\nu]} - i\left[\hat{A}_\mu\,,\,\hat{A}_\nu\right]_\star \tag{3.14}$$

[3] The concept of universal enveloping algebra is reviewed at appendix (C)

[4] For a $\mathcal{U}(L)$ there is nothing like the exponential map that maps a Lie algebra L to a Lie group.

with the gauge transformations

$$\hat{\delta}_{\hat{\Lambda}} \hat{D}_\mu \hat{\psi} = i\hat{\Lambda}(x) \star \hat{D}_\mu \hat{\psi} \tag{3.15}$$

$$\hat{\delta}_{\hat{\Lambda}} \hat{F}_{\mu\nu} = i\left[\hat{\Lambda}(x), \hat{F}_{\mu\nu}\right]_\star . \tag{3.16}$$

The gauge invariant action for the gauge sector is defined by

$$S\left[\hat{A}, \partial\hat{A}\right] = -\frac{1}{4}\int d^D x \,\mathrm{Tr}\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right) \tag{3.17}$$

and the action for the matter/interaction sector is constructed by using the covariant derivative. For example, the action of a NC fermion is written as

$$S\left[\hat{\psi}, \partial\hat{\psi}, \hat{A}\right] = \int d^d x\, \bar{\hat{\psi}} \star (\gamma^\mu \hat{D}_\mu - m)\hat{\psi}. \tag{3.18}$$

These definitions are similar to corresponding commutative $su(n)$ gauge theory, the differences being the ordinary point-wise product and the Lie algebra valued fields and gauge transformation parameters. Here we denote the commutative concepts without the hats: ψ, A_μ, Λ etc. In order to fix the notation we mention that in the commutative space, the fields transform under gauge transformations with Lie algebra-valued infinitesimal parameters

$$\delta_\Lambda \psi(x) = i\Lambda(x)\psi(x) \quad ; \quad \Lambda(x) = \Lambda_a T^a. \tag{3.19}$$

The commutator of two gauge transformations gives us

$$\left(\delta_\Lambda \delta_\Sigma - \delta_\Sigma \delta_\Lambda\right)\psi(x) = i\Lambda_a(x)\Sigma_b(x) f_{abc} T^c \psi(x) = \delta_{\Lambda\times\Sigma}\psi(x), \tag{3.20}$$

where

$$\Lambda \times \Sigma \equiv \Lambda_a \Sigma_b f_{abc} T_c = -i\left[\Lambda\, ,\, \Sigma\right]. \tag{3.21}$$

For the Lie algebra-valued gauge potential $A_{a\mu}(x)$ we define the following transformation

$$\delta_\Lambda A_{a\mu} = \partial_\mu \Lambda_a - f_{abc}\Lambda_b(x) A_{c\mu}(x) \quad ; \quad A_\mu = A_{a\mu}(x) T_a. \tag{3.22}$$

Since the gauge invariance of the commutative gauge theory should be maintained in the NC space, the gauge transformations in the latter theory are induced by the transformations of the former theory:

$$\hat{A}_\mu[A] + \hat{\delta}_{\hat{\Lambda}[\Lambda,A]} \hat{A}_\mu[A] = \hat{A}_\mu[A + \delta_\Lambda A], \tag{3.23}$$

$$\hat{\psi}[\psi, A] + \hat{\delta}_{\hat{\Lambda}[\Lambda,A]} \hat{\psi}[\psi, A] = \hat{\psi}[\psi + \delta_\Lambda \psi\, ,\, A + \delta_\Lambda A]. \tag{3.24}$$

These relations are called SW map. They say that, if the commutative fields A_μ and ψ are related to the fields A_μ^U and ψ^U through the gauge transformation $U = \exp(i\Lambda)$ generated by Λ, then the NC fields $\hat{A}_\mu[A]$ and $\hat{\psi}[\psi, A]$ are related to the fields $\hat{A}_\mu[A^U]$ and $\hat{\psi}[\psi^U, A^U]$

through the gauge transformation $\hat{U} = \exp(i\hat{\Lambda}[\Lambda, A])$, generated by $\hat{\Lambda}[\Lambda, A]$. These gauge equivalence relations can be solved pertubatively in θ in order to obtain the SW maps explicitly. For the gauge theories with $U(N)$ as the gauge group the SW map for the leading order in θ can be written as:

$$\begin{aligned}
\hat{A}_\mu\left[A\right] &= A_\mu + \frac{1}{4}\theta^{\nu\rho}\{A_\rho\,,\ \partial_\nu A_\mu + F_{\mu\nu}\} + \mathcal{O}\left(\theta^2\right) && (3.25)\\
\hat{\psi}\left[\psi, A\right] &= \psi + \frac{1}{2}\theta^{\mu\nu}\rho_\psi(A_\nu)\partial_\mu\psi + \frac{i}{8}\theta^{\mu\nu}\left[\rho_\psi(A_\mu)\,,\ \rho_\psi(A_\nu)\right]\psi + \mathcal{O}\left(\theta^2\right) && (3.26)\\
\hat{\Lambda}\left[\Lambda, A\right] &= \Lambda + \frac{1}{4}\theta^{\mu\nu}\{A_\nu\,,\ \partial_\mu\Lambda\} + \mathcal{O}\left(\theta^2\right). && (3.27)
\end{aligned}$$

As we have mentioned above, the gauge parameters of a general gauge theory, for example, with $SU(N)$ as the gauge group, in the NC space can not be Lie algebra-valued, because the commutation relation is not always closed, they have to take value in enveloping algebra[5].

$$\begin{aligned}
\hat{\Lambda}(x) &= \hat{\Lambda}_a(x)T^a + \hat{\Lambda}^1_{ab}(x) : T^aT^a : + \ldots\\
&+ \hat{\Lambda}^{n-1}_{a_1a_2\ldots a_n}(x) : T^{a_1}\cdots T^{a_n} : + \ldots
\end{aligned}$$

The dots mean that we must sum over a basis of vector space spanned by homogeneous polynomials of the generators of the Lie algebra. Completely symmetrized products form such the following basis:

$$\begin{aligned}
: T^a : &= T^a\\
: T^aT^b : &= \frac{1}{2}\left\{T^a, T^b\right\} = \frac{1}{2}\left(T^aT^b + T^bT^a\right)\\
: T^{a_1}\ldots T^{a_n} : &= \frac{1}{n!}\sum_{\pi\epsilon S_n} T^{a_{\pi(1)}}\cdots T^{a_{\pi(n)}}.
\end{aligned}$$

The $\star$-commutator of two enveloping algebra-valued transformations always will remain enveloping algebra-valued. The bad point is that we will deal with a series of infinite parameters, however it is possible to define a gauge transformation where all these infinitely parameters depend on the usual gauge parameter $\Lambda(x)$, the gauge potential $A_\mu(x)$ and their derivatives [37]. Transformations of this type will be denoted as $\hat{\Lambda}\left[A\right]$ and their x-dependence is purely via this finite set of parameters and gauge potentials $\Lambda\left[A\right] \equiv \hat{\Lambda}\left[A(x)\right]$ (for constant θ).

Now the gauge transformation (3.10) will take the following form

$$\delta_{\hat{\Lambda}}\hat{\psi}(x) = i\hat{\Lambda}\left[A\right] \star \hat{\psi}(x). \tag{3.28}$$

5 As mentioned above just in the case of $U(N)$ gauge group one find that the commutation is closed and the parameters are Lie algebra-valued.

Each finite set of parameters $\Lambda_a^0(x)$ defines a tower $\Lambda_{\Lambda^0}\left[A^0\right]$ in the enveloping algebra that is completely determined by the Lie algebra-valued part. To define and construct this tower we demand a similarity with Lie algebra [81]

$$\left(\delta_{\hat{\Lambda}}\delta_{\hat{\Sigma}} - \delta_{\hat{\Sigma}}\delta_{\hat{\Lambda}}\right)\hat{\psi}(x) = \delta_{\hat{\Lambda}\times\hat{\Sigma}}\hat{\psi}(x). \tag{3.29}$$

More explicitly we have

$$i\delta_{\hat{\Lambda}}\hat{\Sigma}\left[A\right] - i\delta_{\hat{\Sigma}}\hat{\Lambda}\left[A\right] + \hat{\Lambda}\left[A\right]\star\hat{\Sigma}\left[A\right] - \hat{\Sigma}\left[A\right]\star\hat{\Lambda}\left[A\right] = i\hat{\Omega}_{\hat{\Lambda}\times\hat{\Sigma}}\left[A\right]. \tag{3.30}$$

Now we can use the expansion of the $\star$-product to solve Eq.(3.30) in its NC part.

$$\begin{aligned}(f\star g)(x) &= \exp\left(\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}\right) f(x)g(y)|_{y\to x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f\partial_j g + \cdots .\end{aligned}$$

We assume that always the following expansion is possible:

$$\hat{\Lambda}\left[A\right] = \Lambda + \Lambda^1\left[A\right] + \Lambda^2\left[A\right] + \cdots . \tag{3.31}$$

This expansion is the principal ingredient for the construction of non-Abelian NC gauge theories. If we substitute the above relation in (3.30) to zeroth order we yield the Eq.(3.20) which is the commutator of two Lie algebra-valued objects. To the first order by means of an ansatz we have that

$$\Lambda^1\left[A\right] = \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu\Lambda, A_\nu\right\} = \frac{1}{2}\theta^{\mu\nu}\partial_\mu\Lambda_a A_{b\nu} : T^a T^b : . \tag{3.32}$$

Also we can expand the fields, gauge potential and in NC space in terms of the original space ones as follows

$$\hat{\psi} = \psi^0 + \psi^1 + ... \tag{3.33}$$

and

$$\hat{A}_\mu = A_\mu + A_\mu^1 + ... \tag{3.34}$$

By the same treatment as the gauge parameter for the gauge potential and field strength at the first order terms one finds [37]

$$A_k^1 = -\frac{1}{4}\theta^{ij}\left\{A_i, \partial_j A_k + F_{jk}\right\} \tag{3.35}$$

$$F_{ij}^1 = \frac{1}{2}\theta^{kl}\left\{F_{ik}, F_{jl}\right\} - \frac{1}{4}\theta^{kl}\left\{A_k, \left(\partial_l + D_l\right)F_{ij}\right\} \tag{3.36}$$

Hence the ordinary Yang-Mills term $F_{ij}F^{ij}$ in the NC spacetime takes the following form

$$\begin{aligned}\hat{F}_{ij}\star\hat{F}^{ij} &= F_{ij}F^{ij} + \frac{i}{2}\theta^{kl}D_k F_{ij}D_l F^{ij} + \frac{1}{2}\theta^{kl}\{\{F_{ik}, F_{jl}\}, F^{ij}\} \\ &- \frac{1}{4}\theta^{kl}\{F_{kl}, F_{ij}F^{ij}\} - \frac{i}{4}\theta^{kl}\left[A_k, \{A_l, F_{ij}F^{ij}\}\right]. \end{aligned} \tag{3.37}$$

For matter field in the fundamental representation we have

$$\psi^1 = -\frac{1}{4}\theta^{ij} A_i \left(\partial_j + D_j\right)\psi \quad \text{where} \quad D_i\psi = \partial_i\psi - iA_i\psi \tag{3.38}$$

and in the adjoint representation [38]

$$\psi^1 = -\frac{1}{4}\theta^{ij} \left\{A_i, \left(\partial_j + D_j\right)\psi\right\} \quad \text{where} \quad D_i\psi = \partial_i\psi - i\left[A_i, \psi\right] \tag{3.39}$$

We must take care that these variables do not take value in a Lie algebra but in an enveloping algebra. So $\{\bullet, \bullet\}$ is not the anticommutator of a Lie algebra-valued matrices and the result is more complicated such as (3.32).

The higher order of expansions are obtained analogously. In [37] the action of a NC gauge theory with fermionic matter has been constructed to the second order of NCy parameter θ, and the result can be written solely in terms of the usual gauge covariant derivatives and field strengths, exhibits beautifully the usual gauge invariance of the expansion.

3.2.2 The no-go theorem

In a realistic physical model we need to consider gauge groups with several simple factors. Let G_1 and G_2 be two local gauge groups. The gauge group $G = G_1 \times G_2$ is defined by

$$\begin{gathered} g = g_1 \times g_2 \quad ; \quad h = h_1 \times h_2 \quad ; \quad g, h \in G \quad ; \quad g_i, h_i \in G_i \\ g.h = \left(g_1 \times g_2\right).\left(h_1 \times h_2\right) \equiv \left(g_1.h_1\right) \times \left(g_2.h_2\right). \end{gathered} \tag{3.40}$$

where "$\cdot$" is the corresponding group multiplication for each group. If we now take the groups to be the NC ones, $G_1 = U_\star(n)$ and $G_2 = U_\star(m)$, we see that because of the $\star$-product we cannot re-arrange the elements of the subgroups as in (3.40). Therefore the matter fields cannot be in the fundamental representation of both $U_\star(n)$ and $U_\star(m)$. However, there is one possibility left. The matter field Ψ can be in the fundamental representation of one group, say $U_\star(n)$, and in the anti-fundamental representation of the other group

$$\Psi \longrightarrow \Psi' = U \star \Psi \star V^{-1} \quad ; \quad U \in U_\star(n), V \in U_\star(m). \tag{3.41}$$

In the general case the gauge group consists of N factors $G = \prod_{i=1}^{N} U_\star(n_i)$. The matter fields can at most be charged under two of the $U_\star(n_i)$ factors and they have to be singlets under the rest of them. This is a strong constraint on the possible models specially the extension of the standard model of particle physics on NC spacetimes.

4 Doplicher–Fredenhagen–Roberts noncommutative phase-space

4.1 The Noncommutative Quantum Mechanics

In this section we will introduce DFR space and its complete extension formulated in [77, 82, 83, 84, 85] where the beginning version of the DFR is accomplished through the introduction of the canonical conjugate momenta to the variable $\hat{\theta}^{\mu\nu}$ of the system. Concerning now the DFR-extended space, we continue to furnish its "missing parts" and naturally its implications in QM and QFT.

4.1.1 The Snyder's Algebra

In the beginning of the formulation of QFT and the appearance of the notorious divergences, many people were thinking that the problem is caused by the existence of a continuum spacetime and this continuity was dictated by Lorentz invariance. Until Snyder proposed that the Lorentz invariance does not imply continuity necessarily. In his work [13], Snyder introduced a five dimensional spacetime with $SO(4,1)$ as a symmetry group, with generators $\mathbf{M}^{AB}$, satisfying the Lorentz algebra in the 5D De Sitter space, where $A, B = 0, 1, 2, 3, 4$ and using natural units, i.e., $\hbar = c = 1$. Snyder's representation of this algebra was constructed by considering a (4+1)D spacetime with coordinates η_A and metric $g_{AB} = \text{diag}(+----)$. The transformations that leave both η_4 and the quadratic form $\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$ invariant are the Lorentz transformations on this space. The ordinary Lorentz transformations act only on the first four coordinates and are induced by a dimensional reduction from (4+1) to (3+1) dimensions. Snyder defined the generators of usual Lorentz algebra in (3+1)D as

$$\mathbf{M}^{\mu\nu} = i\left(\eta^{\mu}\frac{\partial}{\partial\eta_{\nu}} - \eta^{\nu}\frac{\partial}{\partial\eta_{\mu}}\right). \tag{4.1}$$

Also he introduced the position and time operators in Minkowski spacetime in the following way

$$\begin{aligned} \hat{x}^{\mu} &= a\mathbf{M}^{4\mu} \\ &= ia\left(\eta_4\frac{\partial}{\partial\eta_{\mu}} + \eta^{\mu}\frac{\partial}{\partial\eta_4}\right). \end{aligned} \tag{4.2}$$

In the above definition, the spacetime coordinates are promoted to operators. As it can be seen, the position operators are Hermitian and can be shown that they have discrete spectrum with eigenvalues ma where m are integers. But the time operator is not Hermitian and it also has a continuous spectrum (where $\mu, \nu = 0, 1, 2, 3$ and the parameter a has dimension of length). The mentioned relationship introduces the commutator,

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = ia^2\mathbf{M}^{\mu\nu}, \tag{4.3}$$

and the identities,

$$[\mathbf{M}^{\mu\nu}, \hat{x}^{\lambda}] = i\Big(\hat{x}^{\mu}\eta^{\nu\lambda} - \hat{x}^{\nu}\eta^{\mu\lambda}\Big) \tag{4.4}$$

and

$$[\mathbf{M}^{\mu\nu}, \mathbf{M}^{\alpha\beta}] = i\Big(\mathbf{M}^{\mu\beta}\eta^{\nu\alpha} - \mathbf{M}^{\mu\alpha}\eta^{\nu\beta} + \mathbf{M}^{\nu\alpha}\eta^{\mu\beta} - \mathbf{M}^{\nu\beta}\eta^{\mu\alpha}\Big), \tag{4.5}$$

which agree with four dimensional Lorentz invariance.
We here note that the triple commutator in Snyder's quantized spacetime is not vanishing,

$$[\hat{x}^{\mu}, [\hat{x}^{\nu}, \hat{x}^{\rho}]] = -a^2(\eta^{\nu\mu}\hat{x}^{\rho} - \eta^{\rho\mu}\hat{x}^{\nu}). \tag{4.6}$$

Such a q-number triple commutator is not a general feature of a Lorentz-invariant NC space-time.

We can also construct the Snyder's spacetime algebra conveniently as a modification of the canonical commutation relations of phase-space, given by [86]

$$\begin{aligned} [\hat{x}^{\mu}, \hat{x}^{\nu}] &= ia^2\hbar^{-1}(\hat{x}^{\mu}\hat{p}^{\nu} - \hat{x}^{\nu}\hat{p}^{\mu}) \\ [\hat{x}^{\mu}, \hat{p}_{\nu}] &= i\hbar\delta^{\mu}_{\nu} + ia^2\hbar^{-1}\hat{p}^{\mu}\hat{p}_{\nu} \\ [\hat{p}_{\mu}, \hat{p}_{\nu}] &= 0, \end{aligned} \tag{4.7}$$

where we can see the presence of a fundamental minimal length a, the scale of NCy. In this way we can recover the "usual" phase space of quantum mechanics when $a = 0$.

In fact there are two kinds of Snyder's quantized spacetime where the triple commutator between the operator coordinates does not vanish. In one of them, which was originally proposed by Snyder and was mentioned above, the spatial coordinates have a discrete spectrum of eigenvalues of the form ma, where m is an integer, while the time coordinate has a continuous spectrum. The other one is the opposite: the spectrum of the time coordinate is discrete, while that of the spatial coordinates is continuous [87].

4.1.2 The Doplicher–Fredenhagen–Roberts–Amorim (DFR-extended) Space

The Doplicher, Fredenhagen and Roberts (DFR) algebra [69] essentially *defines*

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hat{\theta}^{\mu\nu} \tag{4.8}$$

as well as the vanishing of the triple commutator among the coordinate operators,

$$[\hat{x}^{\mu}, [\hat{x}^{\nu}, \hat{x}^{\rho}]] = 0, \tag{4.9}$$

and it is easy to realize that this relationship constitutes a constraint in a NC spacetime. Notice that the commutator inside the triple one is not a c-number.

As usual $\hat{x}^{\mu}$ and $\hat{p}_{\nu}$, where $i, j = 1, 2, \ldots, D$ and $\mu, \nu = 0, 1, \ldots, D$, represent the position operator and its conjugate momentum. The NC variable $\hat{\theta}^{\mu\nu}$ represents the NCy operator, but now $\hat{\pi}_{\mu\nu}$ is its conjugate momentum. In accordance with the discussion above, it follows the algebra

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \tag{4.10a}$$
$$[\hat{\theta}^{\mu\nu}, \hat{\pi}_{\alpha\beta}] = i\delta^{\mu\nu}{}_{\alpha\beta}, \tag{4.10b}$$

where $\delta^{\mu\nu}{}_{\alpha\beta} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha$. The relation (4.8) here in a space with D dimensions, for example, can be written as

$$[\hat{x}^i, \hat{x}^j] = i\hat{\theta}^{ij} \qquad \text{and} \qquad [\hat{p}_i, \hat{p}_j] = 0 \tag{4.11}$$

and together with the triple commutator (4.9) condition of the standard spacetime, i.e.,

$$[\hat{x}^\mu, \hat{\theta}^{\nu\alpha}] = 0. \tag{4.12}$$

This implies that

$$[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] = 0, \tag{4.13}$$

and this completes the DFR algebra.

Thus there are two notable differences between Snyder's and the DFR algebras,

$$[\hat{x}^\mu, [\hat{x}^\nu, \hat{x}^\rho]] \quad \begin{cases} = 0 & (\text{ DFR algebra}) \\ \neq 0 & (\text{ Snyder's algebra}) \end{cases}$$
$$[\hat{p}^\mu, \hat{x}^\nu] \quad \begin{cases} = ig^{\mu\nu} & (\text{ DFR algebra}) \\ \neq ig^{\mu\nu} & (\text{ Snyder's algebra}). \end{cases} \tag{4.14}$$

Recently, in order to obtain consistency R. Amorim introduced [77], as we talked above, the canonical conjugate momenta $\hat{\pi}_{\mu\nu}$ such that,

$$[\hat{p}_\mu, \hat{\theta}^{\nu\alpha}] = 0, \qquad [\hat{p}_\mu, \hat{\pi}_{\nu\alpha}] = 0. \tag{4.15}$$

The Jacobi identity formed by the operators $\hat{x}^i$, $\hat{x}^j$ and $\hat{\pi}_{kl}$ leads to the nontrivial relation

$$[[\hat{x}^\mu, \hat{\pi}_{\alpha\beta}], \hat{x}^\nu] - [[\hat{x}^\nu, \hat{\pi}_{\alpha\beta}], \hat{x}^\mu] = -\delta^{\mu\nu}{}_{\alpha\beta}. \tag{4.16}$$

The solution, unless trivial terms, is given by

$$[\hat{x}^\mu, \hat{\pi}_{\alpha\beta}] = -\frac{i}{2}\delta^{\mu\nu}{}_{\alpha\beta}\hat{p}_\nu. \tag{4.17}$$

It is simple to verify that the whole set of commutation relations listed above is indeed consistent under all possible Jacobi identities. Expression (4.17) suggests the shifted coordinate operator [88, 89, 90, 91, 92, 93] (also known as Bopp-shift)

$$\mathbf{X}^\mu \equiv \hat{x}^\mu + \frac{1}{2}\hat{\theta}^{\mu\nu}\hat{p}_\nu \tag{4.18}$$

that commutes with π_{kl}. Actually, (4.18) also commutes with $\hat{\theta}^{kl}$ and $\mathbf{X}^j$, and satisfies a non trivial commutation relation with $\hat{p}_i$ depending objects, which could be derived from

$$[\mathbf{X}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \tag{4.19}$$

and

$$[\mathbf{X}^\mu, \mathbf{X}^\nu] = 0. \tag{4.20}$$

To construct a DFR-extended algebra in (x, θ) space, we can write

$$\mathbf{M}^{\mu\nu} = \mathbf{X}^\mu \hat{p}^\nu - \mathbf{X}^\nu \hat{p}^\mu - \hat{\theta}^{\mu\sigma}\hat{\pi}_\sigma{}^\nu + \hat{\theta}^{\nu\sigma}\hat{\pi}_\sigma{}^\mu, \tag{4.21}$$

where $\mathbf{M}^{\mu\nu}$ is the antisymmetric generator of the Lorentz-group. To construct $\pi_{\mu\nu}$ we have to obey equations (4.10b) and (4.17), obviously. From (4.10a) we can write the generators of translations as

$$P_\mu = -i\partial_\mu. \tag{4.22}$$

With these ingredients it is easy to construct the commutation relations

$$[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \qquad [\mathbf{M}_{\mu\nu}, \mathbf{P}_\rho] = -i\Big(\eta_{\mu\nu}\mathbf{P}_\rho - \eta_{\mu\rho}\mathbf{P}_\nu\Big),$$
$$[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}] = -i\Big(\eta_{\mu\rho}\mathbf{M}_{\nu\sigma} - \eta_{\mu\sigma}\mathbf{M}_{\nu\rho} - \eta_{\nu\rho}\mathbf{M}_{\mu\sigma} - \eta_{\nu\sigma}\mathbf{M}_{\mu\rho}),$$

and we can say that $\mathbf{P}_\mu$ and $\mathbf{M}_{\mu\nu}$ are the generators of the DFR-extended algebra. These relationships are important, because they are essential for the extension of the Dirac equation to the DFR-extended configuration space (x, θ). It can be shown that the Clifford algebra structure generated by the 10 generalized Dirac matrices Γ relies on these relations.

Now we need to remember some basics in quantum mechanics. In order to introduce a continuous basis for a general Hilbert space, with the aid of the above commutation relations, it is necessary firstly to find a maximal set of commuting operators. For instance, let us choose a momentum basis formed by the eigenvectors of $\hat{p}$ and $\hat{\pi}$. A coordinate basis formed by the eigenvectors of $(\mathbf{X}, \hat{\theta})$ can also be introduced, among other possibilities. We observe here that it is in no way possible to form a basis involving more than one component of the original position operator $\hat{x}$, since their components do not commute.

To clarify, let us display the fundamental relations involving those basis, namely eigenvalue, orthogonality and completeness relations

$$\begin{aligned}
&\mathbf{X}^i|X', \hat{\theta}'\rangle = X'^i|X', \hat{\theta}'\rangle, \qquad \hat{\theta}^{ij}|X', \hat{\theta}'\rangle = \theta'^{ij}|X', \hat{\theta}'\rangle,\\
&\hat{p}_i|\hat{p}', \hat{\pi}'\rangle = p'_i|\hat{p}', \hat{\pi}'\rangle, \qquad \hat{\pi}_{ij}|\hat{p}', \hat{\pi}'\rangle = \pi'_{ij}|\hat{p}', \hat{\pi}'\rangle,\\
&\langle X', \hat{\theta}'|X'', \hat{\theta}''\rangle = \delta^D(X' - X'')\delta^{\frac{D(D-1)}{2}}(\hat{\theta}' - \hat{\theta}''),\\
&\langle \hat{p}', \hat{\pi}'|\hat{p}'', \hat{\pi}''\rangle = \delta^D(\hat{p}' - \hat{p}'')\delta^{\frac{D(D-1)}{2}}(\hat{\pi}' - \hat{\pi}''),\\
&\int d^D X'\, d^{\frac{D(D-1)}{2}}\hat{\theta}'|X', \hat{\theta}'\rangle\langle X', \hat{\theta}'| = \mathbf{1},\\
&\int d^D\hat{p}'\, d^{\frac{D(D-1)}{2}}\hat{\pi}'|\hat{p}', \hat{\pi}'\rangle\langle \hat{p}', \hat{\pi}'| = \mathbf{1},
\end{aligned} \tag{4.23}$$

notice that the dimension D means that we live in a framework formed by the spatial coordinates and by the $\hat{\theta}$ coordinates, namely, D includes both spaces, $D =$ (spatial coordinates $+\hat{\theta}$ coordinates). It can be seen clearly from the equations involving the delta functions and the integrals equations in (4.23).

Representations of the operators in those bases can be obtained in an usual way. For instance, the commutation relations given by equations (4.10) to (4.19) and the eigenvalue relations above, unless trivial terms, give

$$\langle X', \hat{\theta}' | \hat{p}_i | X'', \hat{\theta}'' \rangle = -i \frac{\partial}{\partial X'^i} \delta^D (X' - X'') \delta^{\frac{D(D-1)}{2}} (\hat{\theta}' - \hat{\theta}'')$$

and

$$\langle X', \hat{\theta}' | \hat{\pi}_{ij} | X'', \hat{\theta}'' \rangle = -i \delta^D (X' - X'') \frac{\partial}{\partial \hat{\theta}'^{ij}} \delta^{\frac{D(D-1)}{2}} (\hat{\theta}' - \hat{\theta}'').$$

The transformations from one basis to the other one are carried out by extended Fourier transforms. Related with these transformations is the plane wave

$$\langle X', \hat{\theta}' | \hat{p}'', \hat{\pi}'' \rangle = N \exp(i\hat{p}'' \cdot X' + i\hat{\pi}'' \cdot \hat{\theta}'),$$

where internal products are represented in a compact manner. For instance,

$$\hat{p}'' \cdot X' + \hat{\pi}'' \cdot \hat{\theta}' = \hat{p}''_i X'^i + \frac{1}{2} \hat{\pi}''_{ij} \hat{\theta}'^{ij}.$$

Before discussing any dynamics, it seems interesting to study the generators of the group of rotations $SO(D)$. Without considering the spin sector, we realize that the usual angular momentum operator

$$\mathbf{l}^{ij} = \hat{x}^i \hat{p}^j - \hat{x}^j \hat{p}^i$$

does not close in an algebra due to (4.11). And we have that,

$$[\mathbf{l}^{ij}, \mathbf{l}^{kl}] = i\delta^{il}\mathbf{l}^{kj} - i\delta^{jl}\mathbf{l}^{ki} - i\delta^{ik}\mathbf{l}^{lj} + i\delta^{jk}\mathbf{l}^{li} - i\hat{\theta}^{il}\hat{p}^k\hat{p}^j + i\hat{\theta}^{jl}\hat{p}^k\hat{p}^i + i\hat{\theta}^{ik}\hat{p}^l\hat{p}^j - i\hat{\theta}^{jk}\hat{p}^l\hat{p}^i$$

and so their components can not be $SO(D)$ generators in this extended Hilbert space. On the contrary, the operator

$$\mathbf{L}^{ij} := \mathbf{X}^i \hat{p}^j - \mathbf{X}^j \hat{p}^i, \tag{4.24}$$

closes in the $SO(D)$ algebra. However, to properly act in the $(\hat{\theta}, \hat{\pi})$ sector, it has to be generalized to the total angular momentum operator

$$\mathbf{J}^{ij} := \mathbf{L}^{ij} - \hat{\theta}^{il}\hat{\pi}_l{}^j + \hat{\theta}^{jl}\hat{\pi}_l{}^i. \tag{4.25}$$

It is easy to see that not only

$$[\mathbf{J}^{ij}, \mathbf{J}^{kl}] = i\delta^{il}\mathbf{J}^{kj} - i\delta^{jl}\mathbf{J}^{ki} - i\delta^{ik}\mathbf{J}^{lj} + i\delta^{jk}\mathbf{J}^{li}, \tag{4.26}$$

but $\mathbf{J}^{ij}$ generates rotations in all Hilbert space sectors. Actually

$$\delta \mathbf{X}^i = \frac{i}{2}\epsilon_{kl}[\mathbf{X}^i, \mathbf{J}^{kl}] = \epsilon^{ik}\mathbf{X}_k, \qquad \delta \hat{p}^i = \frac{i}{2}\epsilon_{kl}[\hat{p}^i, \mathbf{J}^{kl}] = \epsilon^{ik}\hat{p}_k,$$
$$\delta \hat{\theta}^{ij} = \frac{i}{2}\epsilon_{kl}[\hat{\theta}^{ij}, \mathbf{J}^{kl}] = \epsilon^{ik}\hat{\theta}_k^{\ j} + \epsilon^{jk}\hat{\theta}^i_{\ k}, \qquad \delta \hat{\pi}^{ij} = \frac{i}{2}\epsilon_{kl}[\hat{\pi}^{ij}, \mathbf{J}^{kl}] = \epsilon^{ik}\hat{\pi}_k^{\ j} + \epsilon^{jk}\hat{\pi}^i_{\ k} \tag{4.27}$$

have the expected form. The same occurs with

$$\hat{x}^i = \mathbf{X}^i - \frac{1}{2}\hat{\theta}^{ij}\hat{p}_j \quad \Longrightarrow \quad \delta\hat{x}^i = \frac{i}{2}\epsilon_{kl}[\hat{x}^i, \mathbf{J}^{kl}] = \epsilon^{ik}\hat{x}_k.$$

Observe that in the usual NCQM prescription, where the objects of NCy are parameters or where the angular momentum operator has not been generalized, $\mathbf{X}$ fails to transform as a vector operator under $SO(D)$ [88, 89, 90, 91, 92, 93]. The consistence of transformations (4.27) comes from the fact that they are generated through the action of a symmetry operator and not from operations based on the index structure of those variables.

We would like to mention that in $D = 2$ the operator $\mathbf{J}^{ij}$ reduces to $\mathbf{L}^{ij}$, in accordance with the fact that in this case $\hat{\theta}$ or $\hat{\pi}$ has only one independent component. In $D = 3$, it is possible to represent $\hat{\theta}$ or $\hat{\pi}$ by three vectors and both parts of the angular momentum operator have the same kind of structure, and so the same spectrum. An unexpected addition of angular momentum potentially arises, although the (θ, π) sector can live in a $\mathbf{J} = 0$ Hilbert subspace. Unitary rotations are generated by $U(\omega) = \exp(-i\omega \cdot \mathbf{J})$, while unitary translations, by $T(\lambda, \Xi) = \exp(-i\lambda \cdot \hat{p} - i\Xi \cdot \hat{\pi})$.

4.2 Dynamical Symmetries in NC Theories

In this section we will analyze the dynamical spacetime symmetries in NC relativistic theories by using the DFR-extended algebra depicted in section 2.1. As explained there, the formalism is constructed in an extended spacetime with independent degrees of freedom associated with the object of NCy $\hat{\theta}^{\mu\nu}$. In this framework we can consider theories that are invariant under the Poincaré group $\mathscr{P}$ or under its extension $\mathscr{P}'$. The Noether formalism adapted to such extended $x + \hat{\theta}$ spacetime will be employed.

4.2.1 Coordinate operators and their transformations in relativistic NCQM

In the usual formulations of NCQM, interpreted here as relativistic theories, the coordinates $\hat{x}^\mu$ and their conjugate momenta $\hat{p}_\mu$ are operators acting on a Hilbert space $\mathscr{H}$ satisfying the fundamental commutation relations given in Section 2.1, we can define the operator

$$\mathbf{G}_1 = \frac{1}{2}\omega_{\mu\nu}\mathbf{L}^{\mu\nu}.$$

Note that, analogously to (4.27), it is possible to dynamically generate infinitesimal transformations on any operator $\mathbf{A}$, following the usual rule $\delta\mathbf{A} = i[\mathbf{A}, \mathbf{G}_1]$. For $\mathbf{X}^\mu$, $\hat{p}_\mu$

and $\mathbf{L}^{\mu\nu}$, given in (4.18) and (4.24), with spacetime coordinates, we have the following results

$$\delta\mathbf{X}^{\mu} = \omega^{\mu}{}_{\nu}\mathbf{X}^{\nu}, \qquad \delta\hat{p}_{\mu} = \omega_{\mu}{}^{\nu}\hat{p}_{\nu}, \qquad \delta\mathbf{L}^{\mu\nu} = \omega^{\mu}{}_{\rho}\mathbf{L}^{\rho\nu} + \omega^{\nu}{}_{\rho}\mathbf{L}^{\mu\rho}.$$

However, the physical coordinates fail to transform in the appropriate way. As can be seen, the same rule applied on $\hat{x}^{\mu}$ gives the result

$$\delta\hat{x}^{\mu} = \omega^{\mu}{}_{\nu}\left(\hat{x}^{\nu} + \frac{1}{2}\hat{\theta}^{\nu\rho}\hat{p}_{r}ho\right) - \frac{1}{2}\hat{\theta}^{\mu\nu}\omega_{\nu\rho}\hat{p}^{\rho}, \tag{4.28}$$

which is a consequence of $\hat{\theta}^{\mu\nu}$ not being transformed. Relation (4.28) probably will break Lorentz symmetry in any reasonable theory. The cure for these problems can be obtained by considering $\hat{\theta}^{\mu\nu}$ as an operator in $\mathscr{H}$, and introducing its canonical momentum $\hat{\pi}_{\mu\nu}$ as well. The price to be paid is that $\hat{\theta}^{\mu\nu}$ will have to be associated with extra dimensions, as happens with the formulations appearing in [70, 71, 72, 73, 74, 75].

Moreover, we have that the commutation relation

$$[\hat{x}^{\mu}, \hat{\pi}_{\rho\sigma}] = -\frac{i}{2}\delta^{\mu\nu}{}_{\rho\sigma}\hat{p}_{\nu} \tag{4.29}$$

is necessary for algebraic consistency under Jacobi identities. The set (4.29) completes the algebra displayed in Section 2.1, namely, the DFR-extended algebra. With this algebra in mind, we can generalize the expression for the total angular momentum, equations (4.25) and (4.26).

The framework constructed above permits consistently to write [94, 95]

$$\mathbf{M}^{\mu\nu} = \mathbf{X}^{\mu}\hat{p}^{\nu} - \mathbf{X}^{\nu}\hat{p}^{\mu} - \hat{\theta}^{\mu\sigma}\hat{\pi}_{\sigma}{}^{\nu} + \hat{\theta}^{\nu\sigma}\hat{\pi}_{\sigma}{}^{\mu} \tag{4.30}$$

and consider this object as the generator of the Lorentz group, since it does not only close itself in the appropriate algebra

$$[\mathbf{M}^{\mu\nu}, \mathbf{M}^{\rho\sigma}] = i\eta^{\mu\sigma}\mathbf{M}^{\rho\nu} - i\eta^{\nu\sigma}\mathbf{M}^{\rho\mu} - i\eta^{\mu\rho}\mathbf{M}^{\sigma\nu} + i\eta^{\nu\rho}\mathbf{M}^{\sigma\mu}, \tag{4.31}$$

but it generates the expected Lorentz transformations on the Hilbert space operators. Actually, for $\delta\mathbf{A} = i[\mathbf{A}, \mathbf{G}_2]$, with $\mathbf{G}_2 = \frac{1}{2}\omega_{\mu\nu}\mathbf{M}^{\mu\nu}$, we have that,

$$\begin{gathered}\delta\hat{x}^{\mu} = \omega^{\mu}{}_{\nu}\hat{x}^{\nu}, \qquad \delta\mathbf{X}^{\mu} = \omega^{\mu}{}_{\nu}\mathbf{X}^{\nu}, \qquad \delta\hat{p}_{\mu} = \omega_{\mu}{}^{\nu}\hat{p}_{\nu}, \qquad \delta\hat{\theta}^{\mu\nu} = \omega^{\mu}{}_{\rho}\hat{\theta}^{\rho\nu} + \omega^{\nu}{}_{\rho}\hat{\theta}^{\mu\rho}, \\ \delta\hat{\pi}_{\mu\nu} = \omega_{\mu}{}^{\rho}\hat{\pi}_{\rho\nu} + \omega_{\nu}{}^{\rho}\hat{\pi}_{\mu\rho}, \qquad \delta\mathbf{M}^{\mu\nu} = \omega^{\mu}{}_{\rho}\mathbf{M}^{\rho\nu} + \omega^{\nu}{}_{\rho}\mathbf{M}^{\mu\rho},\end{gathered} \tag{4.32}$$

which in principle should guarantee the Lorentz invariance of a consistent theory. We observe that this construction is possible because of the introduction of the canonical pair $\hat{\theta}^{\mu\nu}$, $\hat{\pi}_{\mu\nu}$ as independent variables. This pair allows the building of an object like $\mathbf{M}^{\mu\nu}$ in (4.30), which generates the transformations given just above dynamically [76] and not merely by taking into account the algebraic index content of the variables.

From the symmetry structure given above, we realize that actually the Lorentz generator (4.30) can be written as the sum of two commuting objects,

$$\mathbf{M}^{\mu\nu} = \mathbf{M}_1^{\mu\nu} + \mathbf{M}_2^{\mu\nu}, \tag{4.33}$$

where

$$\mathbf{M}_1^{\mu\nu} = \mathbf{X}^\mu \hat{p}^\nu - \mathbf{X}^\nu \hat{p}^\mu \qquad \text{and} \qquad \mathbf{M}_2^{\mu\nu} = -\hat{\theta}^{\mu\sigma}\hat{\pi}_\sigma{}^\nu + \hat{\theta}^{\nu\sigma}\hat{\pi}_\sigma{}^\mu, \tag{4.34}$$

as in the usual addition of angular momenta. Of course both operators have to satisfy the Lorentz algebra. It is possible to find convenient representations that reproduce (4.32). In the sector $\mathscr{H}_1$ of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ associated with $(\mathbf{X}, \hat{p})$, it can be used the usual 4×4 matrix representation $D_1(\Lambda) = (\Lambda^\mu{}_\alpha)$, such that, for instance

$$\mathbf{X}'^\mu = \Lambda^\mu{}_\nu \mathbf{X}^\nu. \tag{4.35}$$

For the sector of $\mathscr{H}_2$ relative to $(\hat{\theta}, \hat{\pi})$, it is possible to use the 6×6 antisymmetric product representation

$$D_2(\Lambda) = \left(\Lambda^{[\mu}{}_\alpha \Lambda^{\nu]}{}_\beta\right), \tag{4.36}$$

such that, for instance,

$$\hat{\theta}'^{\mu\nu} = \Lambda^{[\mu}{}_\alpha \Lambda^{\nu]}{}_\beta \hat{\theta}^{\alpha\beta}. \tag{4.37}$$

The complete representation is given by $D = D_1 \oplus D_2$. In the infinitesimal case, $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, and (4.32) are reproduced. There are four Casimir invariant operators in this context and they are given by

$$\mathbf{C}_{j1} = \mathbf{M}_j{}^{\mu\nu}\mathbf{M}_{j\,\mu\nu} \qquad \text{and} \qquad \mathbf{C}_{j2} = \epsilon_{\mu\nu\rho\sigma}\mathbf{M}_j{}^{\mu\nu}\mathbf{M}_j{}^{\rho\sigma}, \tag{4.38}$$

where $j = 1, 2$. We note that although the target space has $10 = 4 + 6$ dimensions, the symmetry group has only 6 independent parameters and not the 45 independent parameters of the Lorentz group in $D = 10$. As we said before, this $D = 10$ spacetime comprises the four spacetime coordinates and the six $\hat{\theta}$ coordinates.

Analyzing the Lorentz symmetry in NCQM following the lines above, permits us to construct the irreducible representations of this symmetry and introduce an appropriate theory, for instance, a scalar or fermion action. We know, however, that the elementary particles are classified according to the eigenvalues of the Casimir operators of the inhomogeneous Lorentz group. Hence, let us extend this approach to the Poincaré group $\mathscr{P}$. By considering the operators presented here, we can in principle consider

$$\mathbf{G}_3 = \frac{1}{2}\omega_{\mu\nu}\mathbf{M}^{\mu\nu} - a^\mu \hat{p}_\mu + \frac{1}{2}b_{\mu\nu}\hat{\pi}^{\mu\nu} \tag{4.39}$$

as the generator of some group $\mathscr{P}'$, which has the Poincaré group as a subgroup. By following the same rule as the one used in the obtainment of (4.32), were $\mathbf{G}_2$ was replaced by $\mathbf{G}_3$, we can arrive at the set of transformations

$$\delta\mathbf{X}^\mu = \omega^\mu{}_\nu \mathbf{X}^\nu + a^\mu, \qquad \delta\hat{p}_\mu = \omega_\mu{}^\nu \hat{p}_\nu, \qquad \delta\hat{\theta}^{\mu\nu} = \omega^\mu{}_\rho \hat{\theta}^{\rho\nu} + \omega^\nu{}_\rho \hat{\theta}^{\mu\rho} + b^{\mu\nu},$$
$$\delta\hat{\pi}_{\mu\nu} = \omega_\mu{}^\rho \hat{\pi}_{\rho\nu} + \omega_\nu{}^\rho \hat{\pi}_{\mu\rho}, \qquad \delta\mathbf{M}_1^{\mu\nu} = \omega^\mu{}_\rho \mathbf{M}_1^{\rho\nu} + \omega^\nu{}_\rho \mathbf{M}_1^{\mu\rho} + a^\mu \hat{p}^\nu - a^\nu \hat{p}^\mu,$$
$$\delta\mathbf{M}_2^{\mu\nu} = \omega^\mu{}_\rho \mathbf{M}_2^{\rho\nu} + \omega^\nu{}_\rho \mathbf{M}_2{}^{\mu\rho} + b^{\mu\rho}\hat{\pi}_\rho{}^\nu + b^{\nu\rho}\hat{\pi}^\mu{}_\rho, \qquad \delta\hat{x}^\mu = \omega^\mu{}_\nu \hat{x}^\nu + a^\mu + \frac{1}{2} b^{\mu\nu}\hat{p}_\nu. \tag{4.40}$$

We observe that there is an unexpected term in the last one of (4.40) system. This is a consequence of the coordinate operator in (4.18), which is a nonlinear combination of operators that act on $\mathscr{H}_1$ and $\mathscr{H}_2$.

The action of $\mathscr{P}'$ over the Hilbert space operators is in some sense equal to the action of the Poincaré group with an additional translation operation on the $(\hat{\theta}^{\mu\nu})$ sector. All its generators close in an algebra under commutation, so $\mathscr{P}'$ is a well defined group of transformations. As a matter of fact, the commutation of two transformations closes in the algebra

$$[\delta_2, \delta_1]\mathbf{y} = \delta_3 \mathbf{y}, \tag{4.41}$$

where $\mathbf{y}$ represents any one of the operators appearing in (4.40). The parameters composition rule is given by

$$\omega^\mu_{3\nu} = \omega^\mu_{1\alpha}\omega^\alpha_{2\nu} - \omega^\mu_{2\alpha}\omega^\alpha_{1\nu}, \qquad a^\mu_3 = \omega^\mu_{1\nu} a^\nu_2 - \omega^\mu_{2\nu} a^\nu_1,$$
$$b^{\mu\nu}_3 = \omega^\mu_{1\rho} b^{\rho\nu}_2 - \omega^\mu_{2\rho} b^{\rho\nu}_1 - \omega^\nu_{1\rho} b^{\rho\mu}_2 + \omega^\nu_{2\rho} b^{\rho\mu}_1. \tag{4.42}$$

4.3 The DFR-extended Harmonic Oscillator

In [78] the authors have analyzed an harmonic oscillator constructed in a DFR-extended [77] phase-space. The generalized Hamiltonian is given by

$$H \;=\; \frac{\pi^2}{2\Lambda} \,+\, \frac{p^2}{2m} \,+\, V(x^i, p_i, \theta^{ij}, \pi_{ij}) \;, \tag{4.43}$$

where Λ is a parameter with (length)$^{-3}$ dimension and the potential V is a function of DFR-extended variables. Let us define the following symplectic variables ξ^i as being $(x^i,\, p_i,\, \theta^{ij},\, \pi_{ij})$.

We can write the generalized Poisson bracket for this system in a compact and symplectic form as

$$\{F, G\} = \left\{\xi^i, \xi^j\right\} \frac{\partial F}{\partial \xi^i}\frac{\partial G}{\partial \xi^j}, \tag{4.44}$$

where we are using the sum rule for repeated indices.

Hence, following (4.44) the equations of motion for $(x^i,\, p_i,\, \theta^{ij},\, \pi_{ij})$ are given by

$$\dot{x}^i = \theta^{ij}\frac{\partial V}{\partial x^j} + \Big(\frac{\partial V}{\partial p^i} + \frac{p^i}{m}\Big) + \Big(\frac{\pi^{ji}}{\Lambda} + \frac{\partial V}{\partial \pi_{ji}}\Big)\,p_j \tag{4.45}$$

$$\dot{p}_i = -\frac{\partial\, V}{\partial\, x^i} \tag{4.46}$$

$$\dot{\theta}^{ij} = \frac{2}{\Lambda}\pi^{ij} + 2\frac{\partial\, V}{\partial\, \pi_{ij}} \tag{4.47}$$

$$\dot{\pi}_{ij} = -\,2\,\frac{\partial\, V}{\partial\, \theta^{ij}} + \frac{\partial\, V}{\partial\, x^i}\,p_j \tag{4.48}$$

and it can be seen clearly that when $\pi_{ij} = 0$ (namely, when the phase-space is (x, p, θ)) the first consequence is that the potential V will not be a function of π_{ij} and to construct Eq. (4.48) makes no sense. The second consequence is that, from Eq. (4.47), when $\pi_{ij} = 0$ we have that θ^{ij} =const., and therefore the Lorentz invariance is lost and we have a canonical NCy. Let us continue with a specific construction for the potential V, for example.

In [77] an isotropic NC harmonic oscillator (NCHO) was constructed in a $D = 9$ DFR-extended phase-space. The extended potential was given by

$$V(x^i, p_i, \theta^{ij}, \pi_{ij}) = \frac{1}{2}m\omega^2\Big(x^i + \frac{1}{2}\theta^{ij}p_j\Big)^2 + \frac{1}{2}\Lambda\Omega^2\theta^2 \tag{4.49}$$

and the extended Hamiltonian can be written as

$$H = \frac{1}{2\Lambda}\pi^2 + \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2\Big(x^i + \frac{1}{2}\theta^{ij}p_j\Big)^2 + \frac{1}{2}\Lambda\Omega^2\theta^2\ . \tag{4.50}$$

Consequently, the equations of motion are

$$\dot{x}^i = \frac{1}{2}\theta^{ij}\Big(m\omega^2 x_j + \frac{1}{2}m\omega^2\theta_{jl}p^l\Big) + \frac{p^i}{m} + \frac{\pi^{ij}}{\Lambda}\,p_j\ , \tag{4.51}$$

$$\dot{p}_i = -\,m\omega^2 x_i - \frac{1}{2}m\omega^2\theta_{ij}\,p^j\ , \tag{4.52}$$

$$\dot{\theta}^{ij} = \frac{2}{\Lambda}\pi^{ij}\ , \tag{4.53}$$

$$\dot{\pi}_{ij} = -\,2\,\Lambda\Omega^2\theta_{ij}\ . \tag{4.54}$$

In a naive way, it should be possible that we could understand that when $\pi_{ij} = 0$ it would be easy to conclude that the resulting phase-space would be given by the DFR one. However, as we mentioned before when we have analyzed the equations of motion for θ^{ij} and π_{ij} in Eqs. (4.47) and (4.48) respectively, we can see that $\theta_{ij} = const.$. If $\pi_{ij} = 0$ in (4.53) we can see clearly that $\theta^{ij} =$ const.. If we construct a Hamiltonian independent of π_{ij} it does not make sense to construct Eqs. (4.48) and (4.54). Substituting these values in Eqs. (4.51) and (4.52) we recover the canonical NCy and not the DFR NCy approach. Consequently we can conclude that the DFR-extended and pure DFR formalisms are both connected to the canonical NCy via π_{ij} and not only via the nature of θ^{ij}. Namely, to carry out a dimensional reduction of the phase-space (doing $\pi_{ij} = 0$) means that θ^{ij} loses automatically its variable parameter identity and becomes again a constant parameter. Hence, the phase-space dimensional reduction would be represented by $(x^i, p_i, \theta^{ij}, \pi_{ij}) \longrightarrow (x^i, p_i)$ where θ^{ij} is only a constant parameter, the result of the operatorial bracket between x's. The Lorentz invariance is lost and the NCy is the canonical one.

So, concerning the original DFR formalism, although in general, the momentum π_{ij} may not be relevant, we understand that the momentum associated to θ^{ij} is necessary. As a matter of fact, it would be natural and direct to construct this object since θ^{ij}, in DFR phase-space, is a coordinate and must have an associated momentum. However, what is new, in our point of view, is to connect the existence of π_{ij} with the kind of the NCy or, in other words, if the NCy is DFR-extended or canonical.

This result make us think that, if we consider, for example, QFT's systems embedded in a NC spacetime, the implications are even more serious because the existence of a $\theta^{\mu\nu}$-variable NC parameter recovers the Lorentz invariance of the NC theory. But, the relevance of $\pi_{\mu\nu} = 0$ is the fact that it brings back a constant $\theta^{\mu\nu}$, and hence we have the Lorentz invariance violated. So, the connection between both objects ($\theta^{\mu\nu}$ and $\pi_{\mu\nu}$) is a connection between Lorentz invariant or non-invariant NC theories. Besides, we will see that the momenta $\pi_{\mu\nu}$ allow us to construct the commutation relations for the scalar field in DFR phase-space.

Back to Eqs. (4.51)-(4.54) we can see that, in this specific example that, from Eq. (4.53), if $\theta =$const. $\Longrightarrow \pi = 0$ and Eq. (4.54) makes no sense at all. Hence, we have the inverse condition, i.e., $\theta =$const. $\Longrightarrow \pi = 0$, which is the inverse of $\pi = 0 \Longrightarrow \theta =$const.. Let us see another example, the NC relativistic particle to reinforce these claims above.

4.4 The NC Relativistic Particle

In [96], the author proposed that the cure for the lack of relativistic invariance for NC models is to modify the constant feature of the NC parameter. Consequently, he has analyzed the NC version for D-dimensional relativistic particle with a θ-variable phase-space and a π-momentum.

Since we are interested in the dynamics of the phase-space, we have calculated the equations of motion and the NC relativistic acceleration in order to discuss the $\theta_{const.} \Longrightarrow \theta_{variable}$ duality and its consequence. We will see that although the algebra is not the DFR one the consequences of the duality are kept, namely, if we have a θ-variable the phase-space must have the π-momentum (the DFR-momentum).

4.4.1 Noncommutative Relativistic Free Particle

In this section, since we are interested in the DFR features that exist in the analyzed model, we will mention only the relavant points and more details can be found in [96].

The Lagrangian of NC free relativistic particle is

$$S=\int d\tau\left[\dot{x}^{\mu}v_{\mu}-\frac{e}{2}\left(v^{2}-m^{2}\right)+\frac{1}{\theta^{2}}\dot{v}_{\mu}\theta^{\mu\nu}v_{\nu}\right]. \tag{4.55}$$

where $\theta^2 \equiv \theta^{\mu\nu}\theta_{\mu\nu}$, $\eta=(+,-,...,-)$, and p^μ, π^μ, p_e, $p_\theta^{\mu\nu}$ are the conjugate momenta associated to $x^\mu(\tau)$, $v^\mu(\tau)$, $e(\tau)$ and $\theta^{\mu\nu}(\tau)$, respectively. We will use the fundamental algebra [96] defined by

$$\begin{aligned}
\{x^{\mu},x^{\nu}\} &= -\frac{2}{\theta^{2}}\,\theta^{\mu}_{\nu}, \qquad \{x^{\mu},p_{\nu}\}=\delta^{\mu}_{\nu}, \qquad \{v^{\mu},\pi_{\nu}\}=\delta^{\mu}_{\nu} \\
\{x^{\mu},v_{\nu}\} &= \delta^{\mu}_{\nu}, \qquad \{x^{\mu},\pi^{\nu}\}=-\frac{1}{\theta^{2}}\theta^{\mu\nu}, \qquad \{\theta_{\mu\nu},p_{\theta}^{\rho\sigma}\}=-\delta^{[\rho}_{\mu}\,\delta^{\sigma]}_{\nu} \\
\{x^{\mu},p_{\theta}^{\rho\sigma}\} &= -\{\pi^{\mu},p_{\theta}^{\rho\sigma}\}=\frac{1}{\theta^{2}}\eta^{\nu[\rho}v^{\sigma]}-\frac{4}{\theta^{4}}(\theta v)^{\mu}\theta^{\rho\sigma}.
\end{aligned} \tag{4.56}$$

This system is singular and has the following primary constraints

$$G^{\mu} = p^{\mu}-v^{\mu} \tag{4.57}$$

$$T^{\mu} = \pi^{\mu}-\frac{1}{\theta^{2}}\theta^{\mu\nu}v_{\nu} \tag{4.58}$$

$$p_{\theta}^{\mu\nu} = 0 \tag{4.59}$$

$$p_{e} = 0 \tag{4.60}$$

and we can write the total Hamiltonian as being

$$H=\frac{e}{2}\left(v^{2}-m^{2}\right)+\lambda_{1\mu}G^{\mu}+\lambda_{2\mu}T^{\mu}+\lambda_{e}p_{e}+\lambda_{\theta\mu\nu}p_{\theta}^{\mu\nu}. \tag{4.61}$$

where the λ's are the Lagrange multipliers. Using the time consistency we have the secondary constraint

$$K\equiv v^{2}-m^{2}=0 \tag{4.62}$$

and other relations that allow us to determine the Lagrange multipliers

$$\dot{G}^{\mu}=\{G^{\mu},H\}=0 \implies \lambda_{2}^{\mu}=0 \tag{4.63}$$

$$\dot{T}^{\mu}=\{T^{\mu},H\}=0 \implies \lambda_{1}^{\mu}=ev^{\mu}+\frac{2}{\theta^{2}}\left(\lambda_{\theta}v\right)^{\mu}-\frac{4}{\theta^{4}}\left(\theta\lambda_{\theta}\right)\left(\theta v\right)^{\mu}\ . \tag{4.64}$$

If we substitute the fixed Lagrange multipliers into the Hamiltonian we have that

$$H=\frac{e}{2}\left(p^{2}-m^{2}\right)+\left(ev^{\mu}+\frac{2}{\theta^{2}}\left(\lambda_{\theta}v\right)^{\mu}-\frac{4}{\theta^{4}}\left(\theta\lambda_{\theta}\right)\left(\theta v\right)^{\mu}\right)\left(p^{\mu}-v^{\mu}\right)+\lambda_{e}p_{e}+\lambda_{\theta\mu\nu}p_{\theta}^{\mu\nu} \tag{4.65}$$

and it can be seen we were left with two undetermined Lagrange multipliers.

In the same way as we have carried out to construct Eq. (4.44) we will define the following symplectic variables

$$
\begin{aligned}
\xi^{\mu} &\equiv (x^{\mu}, p_{\mu}) \\
\zeta^{\mu} &\equiv (v^{\mu}, \pi_{\mu}) \\
\chi^{\mu} &\equiv (e, p_{e}) \\
\Omega^{\mu\nu} &\equiv (\theta^{\mu\nu}, p_{\theta\mu\nu})
\end{aligned}
\tag{4.66}
$$

We can write the Poisson brackets for this system in a compact and symplectic form as follow

$$
\begin{aligned}
\{F, G\} &= \{\xi^{\mu}, \xi^{\nu}\} \frac{\partial F}{\partial \xi^{\mu}} \frac{\partial G}{\partial \xi^{\nu}} + \{\zeta^{\mu}, \zeta^{\nu}\} \frac{\partial F}{\partial \zeta^{\mu}} \frac{\partial G}{\partial \zeta^{\nu}} \\
&+ \{\chi^{\mu}, \chi^{\nu}\} \frac{\partial F}{\partial \chi^{\mu}} \frac{\partial G}{\partial \chi^{\nu}} + \{\Omega^{\mu\nu}, \Omega^{\rho\sigma}\} \frac{\partial F}{\partial \Omega^{\mu\nu}} \frac{\partial G}{\partial \Omega^{\rho\sigma}} \; .
\end{aligned}
\tag{4.67}
$$

According to the (4.67) we obtain the following equation of motion for x^{μ} and p^{μ}

$$
\begin{aligned}
\dot{x}^{\mu} &= \{x^{\mu}, H\} \\
&= \{x^{\alpha}, p_{\beta}\} \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{\partial H}{\partial p_{\beta}} + \{p_{\beta}, x^{\alpha}\} \frac{\partial x^{\mu}}{\partial p_{\beta}} \frac{\partial H}{\partial x^{\alpha}} + \left\{x^{\alpha}, x^{\beta}\right\} \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{\partial H}{\partial x^{\beta}} \\
&+ \{x^{\alpha}, p_{\theta\rho\sigma}\} \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{\partial H}{\partial p_{\theta\rho\sigma}} + \{p_{\theta\rho\sigma}, x^{\alpha}\} \frac{\partial x^{\mu}}{\partial p_{\theta\rho\sigma}} \frac{\partial H}{\partial x^{\alpha}}
\end{aligned}
\tag{4.68}
$$

$$
\Rightarrow \dot{x}^{\mu} = e p^{\mu} + \frac{2}{\theta^2} (\lambda_{\theta} v)^{\mu} - \frac{4}{\theta^4} (\theta \lambda_{\theta}) (\theta v)^{\mu}
\tag{4.69}
$$

and for p_{μ} we have that

$$
\begin{aligned}
\dot{p}^{\mu} &= \{p^{\mu}, H\} \\
&= \{x^{\alpha}, p_{\beta}\} \frac{\partial p^{\mu}}{\partial x^{\alpha}} \frac{\partial H}{\partial p_{\beta}} + \{p_{\beta}, x^{\alpha}\} \frac{\partial p^{\mu}}{\partial p_{\beta}} \frac{\partial H}{\partial x^{\alpha}} \\
\Rightarrow \dot{p}^{\mu} &= 0
\end{aligned}
\tag{4.70}
$$

Analogously, we can compute the equations of motion for the other variables, namely,

$$
\dot{\theta}^{\mu\nu} = -2\, \lambda_{\theta}^{\mu\nu} \tag{4.71}
$$

$$
\dot{v}_{\mu} = 0 \tag{4.72}
$$

$$
\dot{e} = \lambda_{e} \tag{4.73}
$$

$$
\dot{p}_{e} = -v \cdot p + \frac{1}{2}(v^2 + m^2) \tag{4.74}
$$

$$
\dot{\pi}^{\mu} = \frac{4}{\theta^4} (\theta \lambda_{\theta}) (\theta\, v)^{\mu} - \frac{1}{\theta^2} \eta^{\mu[\rho} v^{\sigma]}\, \lambda_{\theta\rho\sigma} \tag{4.75}
$$

$$
\dot{p}_{\theta}^{\mu\nu} = \frac{8}{\theta^4} \left[\frac{\theta^{\mu\nu}}{\theta^2} (\theta \lambda_{\theta}) (\theta v)^{\sigma} p_{\sigma} - \lambda_{\theta}^{\mu\nu} (\theta v)^{\sigma} p_{\sigma} - \theta^{\mu\nu} (\lambda_{\theta} v)^{\sigma} p_{\sigma} + \frac{1}{2} (\theta \lambda_{\theta}) v^{[\mu} p^{\nu]} \right] \tag{4.76}
$$

Finally, in the same way we can calculate the acceleration in this NC phase-space, namely, $\ddot{x}^\mu = \{\dot{x}^\mu, H\}$, which brings us the result

$$\ddot{x}^\mu = \frac{8}{\theta^4}\Big[(\theta\lambda_\theta)(\lambda_\theta v)^\mu - \frac{4}{\theta^2}(\theta\lambda_\theta)^2(\theta v)^\mu + \lambda^2(\theta v)^\mu - (\theta\lambda_\theta)(\lambda_\theta v)^\mu\Big] \tag{4.77}$$

where $\lambda^2 = \lambda_{\theta\mu\nu}\lambda_\theta^{\mu\nu}$. This last result is very interesting since the equation of motion (4.71) shows us that if we have that $\theta = $ const., we have that $\lambda_\theta = 0$. In this way we will not have p_θ in the Hamiltonian written in (4.65). However, we can easily see from Eq. (4.76) that we have that $\lambda_\theta = 0 \Longrightarrow \dot{p}_\theta = 0 \Longrightarrow p_\theta = $ const., but the important fact is that the phase-space for the Hamiltonian in Eq. (4.65) will not have p_θ. Hence, although the algebra in Eq. (4.56) is not a DFR* one, the scenario is the same, namely, if θ is not constant, the NC phase-space contains p_θ, if θ is constant, we do not have p_θ within the phase-space. Notice that although the λ's are auxiliary variables in order to construct the total Hamiltonian, they are connected to the momenta, by construction of the constraints formalism.

We can also notice that if $\theta = $ const. in Eq. (4.77), the acceleration is zero. This is an interesting result since we do not have any time derivative of θ in Eq. (4.77) but this result is a consequence of the zeroness of λ_θ. However, the time derivative of x^μ in Eq. (4.68) is not zero when $\lambda_\theta = 0$ neither it is constant since $e(\tau)$ is variable (p_μ is constant since $\dot{p}_\mu = 0$).

4.5 Quantum NC scalar field theory

In this section we will construct the first basic step of a QFT with the phase-space definitions established in the previous sections. Since we have shown that the DFR and DFR-extended phase-space are in fact the same, we will use the name DFR to define the formalism embedded in the complete phase-space (x, p, θ, π).

In a series of papers [97, 98, 99], the authors have shown that the construction of the commutation relations between the bosonic/fermionic fields with themselves and with its associated momenta are missing. It is our intention in this section to fill this gap. In other words, we will demonstrate precisely the basic commutation relations using only the DFR elements. The fermionic construction is an ongoing research that will be published in a near future.

In some papers that considers the DFR formalism or a kind of it, such as [69, 70, 73, 74, 75, 96, 72, 100] for example, we can find this basic step in an indirect way where the associated momenta are not defined. The quantity used to construct the scalar field, that was used to associate with the variable θ, is a scalar quantity with no definition at all. As a consequence we now know that this last object is in fact the momenta associated with the NC parameter and this fact allows us to work with a well defined phase-space, the DFR one.

After the considerations given above, we can complement (clarify) [70, 75] by constructing the Fourier transform, so we can write a map between a member of the operator algebra and an ordinary function

$$\hat{f}(\hat{x},\hat{\theta}) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{d^6\pi}{(2\pi)^6}\; e^{-i(p\cdot\hat{x}\,+\,\pi\cdot\hat{\theta})}\;\tilde{f}(p,\pi)\;, \tag{4.78}$$

The Fourier transform $\tilde{f}$ is defined by the trace calculus

$$\tilde{f}(p,\pi) = \mathrm{Tr}\left[e^{i\left(p\cdot\hat{x}\,+\,\pi\cdot\hat{\theta}\right)}\;\hat{f}(\hat{x},\hat{\theta})\right] = \int d^4x\, d^6\theta\; e^{i(p\cdot x\,+\,\pi\cdot\theta)}\; f(x,\theta)\;, \tag{4.79}$$

where $p\cdot\hat{x} = p_\mu\hat{x}^\mu$ and $\pi\cdot\hat{\theta} = \frac{1}{2}\,\pi_{\mu\nu}\hat{\theta}^{\mu\nu}$ (the 1/2 factor avoids the sum over repeated terms), $f(x,\theta)$ is the correspondent function to the operator $\hat{f}(\hat{x},\hat{\theta})$, and the integration measures are

$$\begin{aligned} d^6\pi \;&=\; d\pi_{01}\, d\pi_{02}\, d\pi_{03}\, d\pi_{12}\, d\pi_{13}\, d\pi_{23}\;, \\ d^6\theta \;&=\; d\theta^{01} d\theta^{02} d\theta^{03} d\theta^{12} d\theta^{13} d\theta^{23}\;. \end{aligned} \tag{4.80}$$

The details about θ and π are described in [85]. But notice that in [85] (and references therein), θ-variable and π are not necessarily connected as we have discussed so far. It is important to say that we have clarified the one other main point treated in [70] and [75]. In these last ones, the objects were described with a not well defined quantity coupled to $\theta^{\mu\nu}$. Here we have demonstrated precisely that this quantity is the momentum π which completes the DFR phase-space.

Since Eqs. (4.11), (4.12) and (4.13) closes the extended DFR algebra, we can use the fact that the momentum $\pi_{\mu\nu}$ makes part of the NC phase-space, let us construct the operator field in this DFR algebra in Weyl representation [75]

$$\hat{\phi}(\hat{x},\hat{\theta}) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{d^6\pi}{(2\pi)^6}\;\tilde{\phi}(p,\pi)\; e^{i(p\cdot\hat{x}+\pi\cdot\hat{\theta})}\;, \tag{4.81}$$

where $\tilde{\phi}(p,\pi)$ is the Fourier transform of $\hat{\phi}(\hat{x},\hat{\theta})$ and $d^6\pi$ is a Lorentz invariant measure given above. Notice that the difference between the issues explored here and in [75] is that now we know that the phase-space is described by (x,p,θ,π).

In order to obtain a Fourier representation of a scalar ϕ from operator $\hat{\phi}$, let us make the diagonalization operation [75]

$$\phi(x,\theta) = \langle X,\theta|\,\hat{\phi}(\hat{x},\hat{\theta})\,|X,\theta\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{d^6\pi}{(2\pi)^6}\;\tilde{\phi}(p,\pi)\; e^{i(p\cdot x+\pi\cdot\theta)}\;, \tag{4.82}$$

where we have used that $p\cdot X = p\cdot x$.

The Lagrangian density of a real spin-0 field ϕ with mass m can be written as [85]

$$\mathcal{L} \;=\; \frac{1}{2}\,\partial_\mu\phi\star\partial^\mu\phi \;+\; \frac{\lambda^2}{4}\,\partial_{\mu\nu}\phi\star\partial^{\mu\nu}\phi \;-\; \frac{1}{2}\,m^2\,\phi\star\phi\;, \tag{4.83}$$

where $\partial_{\mu\nu} := \frac{\partial}{\partial\theta^{\mu\nu}}$, and λ is a parameter with dimension of length, as the Planck length which was introduced here due to dimensional needs. It is important to remember that when the Lagrangian (4.83) is integrated throughout DFR space-time, the Moyal product in the quadratic terms reduces to usual product. Therefore, the action from (4.83) gives us the Klein-Gordon equation

$$\left(\Box \,+\, \lambda^2 \Box_\theta \,+\, m^2\right) \phi \,=\, 0 \,, \tag{4.84}$$

where $\Box = \partial_\mu \partial^\mu$ and $\Box_\theta \,=\, \frac{1}{2}\,\partial_{\mu\nu}\partial^{\mu\nu}$ is the four- and six-dimensional Laplace operators, respectively. The canonical conjugate momentum associated to ϕ is given by

$$\pi(x,\theta) \,=\, \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,\theta)} \,=\, \dot{\phi}(x,\theta) \,, \tag{4.85}$$

and this result leads us to the Hamiltonian density

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\,\pi(x,\theta) \star \pi(x,\theta) \,+\, \frac{1}{2}\,\nabla\phi(x,\theta) \star \nabla\phi(x,\theta) \\ +\, \frac{\lambda^2}{2}\,\nabla_\theta\phi(x,\theta) \star \nabla_\theta\phi(x,\theta) \,+\, \frac{1}{2}\,m^2 \phi(x,\theta) \star \phi(x,\theta) \,, \end{aligned} \tag{4.86}$$

where $\nabla_\theta = \frac{1}{2}\,\partial^{ij}$. The conserved field energy is defined by the integral of the Hamiltonian density in the space $(\mathbf{x},\theta)$

$$H \,=\, \int d^3\mathbf{x}\, d^6\theta\, \frac{1}{2}\, \Big[\pi^2(x,\theta) \,+\, (\nabla\phi(x,\theta))^2 \,+\, \lambda^2 \left(\nabla_\theta \hat{\Phi}(x,\theta)\right)^2 \,+\, m^2\, \phi^2(x,\theta)\Big] \,. \tag{4.87}$$

In [75] the author has written an incomplete $\hat{\phi}(x,\theta)$ using the Weyl representation. We say incomplete because now we know that $\theta^{\mu\nu}$ has an associated momentum given by $\pi_{\mu\nu}$. In this way now we can expand the field $\phi(x,\theta)$ with respect to a basis. Let us use the set of plane waves such as

$$u_{\mathbf{p},\pi}(\mathbf{x},\theta) \,=\, N_{\mathbf{p},\pi}\; e^{i\,(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta)} \,, \tag{4.88}$$

which means that we can write the Fourier modes as

$$\phi(\mathbf{x},\theta,t) \,=\, \int d^3\mathbf{p}\;d^6\pi\; N_{\mathbf{p},\pi}\; e^{i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta)}\, a_{\mathbf{p},\pi}(t) \,, \tag{4.89}$$

where $N_{\mathbf{p},\pi}$ is a normalization constant. If we substitute Eq. (4.89) into (4.84) we will have the following equation of motion

$$\ddot{a}_{\mathbf{p},\pi}(t) \,+\, \omega^2_{\mathbf{p},\pi}\, a_{\mathbf{p},\pi}(t) = 0 \,, \tag{4.90}$$

which has a general solution given by

$$a_{\mathbf{p},\pi}(t) \,=\, a^{(1)}_{\mathbf{p},\pi}\; e^{-i\omega_{\mathbf{p},\pi} t} \,+\, a^{(2)}_{\mathbf{p},\pi}\; e^{i\omega_{\mathbf{p},\pi} t} \,, \tag{4.91}$$

and the dispersion relation is

$$\omega_{\mathbf{p},\pi} \;=\; \sqrt{\mathbf{p}^2 \,+\, \frac{\lambda^2}{2}\,\pi^2 \,+\, m^2}\;, \tag{4.92}$$

and from (4.91) we can easily see that $a^{(1)}_{\mathbf{p},\pi}$ and $a^{(2)}_{\mathbf{p},\pi}$ are constants in time. The real-valued feature of the classical field shows us that, of course, the operator is hermitian, hence,

$$\left(a^{(1)}_{\mathbf{p},\pi}\right)^{\dagger} \;=\; a^{(2)}_{-\mathbf{p},-\pi}\;, \tag{4.93}$$

which is a standard constraint. One can ask if the field quanta will obey a kind of Bose-Einstein statistics in this NC phase-space. For now, we will associate $a_{p,\pi}$ and $a^{\dagger}_{p,\pi}$ with annihilation and creation operators, respectively, in DFR formalism. Therefore, the field ϕ in (4.89) is promoted to the field-operator $\hat{\Phi}$ expanded in this basis as

$$\hat{\Phi}(\mathbf{x},\theta,t) \;=\; \int d^3\mathbf{p}\; d^6\pi\; N_{\mathbf{p},\pi}\left[\hat{a}_{\mathbf{p},\pi}\, e^{i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta-\omega_{\mathbf{p},\pi}t)} \,+\, \hat{a}^{\dagger}_{\mathbf{p},\pi}\, e^{-i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta-\omega_{\mathbf{p},\pi}t)}\right]. \tag{4.94}$$

Thus we construct the conjugate momentum operator $\hat{\Pi}$, that is, $\hat{\Pi}(\mathbf{x},\theta,t) \;=\; \dot{\hat{\Phi}}(\mathbf{x},\theta,t)$, so we have that

$$\hat{\Pi}(\mathbf{x},\theta,t) \;=\; \int d^3\mathbf{p}\, d^6\pi\; N_{\mathbf{p},\pi}\,(-i\omega_{\mathbf{p},\pi})\left[\hat{a}_{\mathbf{p},\pi}\, e^{i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta-\omega_{\mathbf{p},\pi}t)} \,-\, \hat{a}^{\dagger}_{\mathbf{p},\pi}\, e^{-i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta-\omega_{\mathbf{p},\pi}t)}\right]. \tag{4.95}$$

The free field can be expanded in terms of creation and annihilation operators, namely,

$$\left[\hat{a}_{\mathbf{p},\pi}\,,\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\right] \;=\; \delta^3(\mathbf{p}-\mathbf{p}')\,\delta^6(\pi-\pi')\;, \tag{4.96}$$

$$\left[\hat{a}_{\mathbf{p},\pi}\,,\,\hat{a}_{\mathbf{p}',\pi'}\right] \;=\; \left[\hat{a}^{\dagger}_{\mathbf{p},\pi}\,,\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\right] \;=\; 0\;. \tag{4.97}$$

We can construct the Moyal commutation relation between two field-operators in equal times as

$$\left[\hat{\Phi}(\mathbf{x},\theta,t)\,,\,\hat{\Phi}(\mathbf{x}'\,,\,\theta',t)\right]_{\star} := \hat{\Phi}(\mathbf{x},\theta,t)\star\hat{\Phi}(\mathbf{x}',\theta',t) - \hat{\Phi}(\mathbf{x}',\theta',t)\star\hat{\Phi}(\mathbf{x},\theta,t) \tag{4.98}$$

and substituting Eq. (4.94) in Eq. (4.98), and using relations (4.96) and (4.97), we obtain

$$\begin{aligned}
\left[\hat{\Phi}(\mathbf{x},\theta,t),\hat{\Phi}(\mathbf{x}',\theta',t)\right]_{\star} &= \int d^9P\int d^9P'\; N_{\mathbf{p},\pi}\,N_{\mathbf{p}',\pi'}\,(-2i)\,\sin\left(\frac{p\wedge p'}{2}\right)\times\\
&\times\big[\,\hat{a}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}t+\pi\cdot\theta+\pi'\cdot\theta')} \,-\, \hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,\hat{a}_{\mathbf{p},\pi}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}t+\pi\cdot\theta-\pi'\cdot\theta')}\\
&-\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}t+\pi\cdot\theta-\pi'\cdot\theta')} \,+\, \hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}t+\pi\cdot\theta+\pi'\cdot\theta')}\big]\,,
\end{aligned} \tag{4.99}$$

where $d^9P := d^3\mathbf{p}\;d^6\pi$, and we have defined the product $p\wedge p' = \theta^{\mu\nu}p_\mu\,p'_\nu$. With the help of the previous calculus, the Moyal-commutation relation between field operator $\hat{\Phi}$ and momenta $\hat{\Pi}$ is given by

$$\Big[\,\hat{\Phi}(\mathbf{x},\theta,t),\hat{\Pi}(\mathbf{x}',\theta^{'},t)\,\Big]_{\star}=\int\,d^9P\,N^2_{\mathbf{p},\pi}\,\,(i\omega_{\mathbf{p},\pi})\,\Big[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+i\pi\cdot(\theta-\theta')}\,+\,e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-i\pi\cdot(\theta-\theta')}\Big]$$
$$+\int d^9P\int d^9P'\,N_{\mathbf{p},\pi}\,N_{\mathbf{p}',\pi'}\,(i\omega_{\mathbf{p}',\pi'})\,(2i)\,\sin\left(\frac{p\wedge p'}{2}\right)\,\times$$
$$\times[\,\hat{a}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}\,+\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,\hat{a}_{\mathbf{p},\pi}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}$$
$$-\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}\,+\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}\,]\,. \tag{4.100}$$

If we choose the normalization constant

$$N_{\mathbf{p},\pi}\;=\;\frac{1}{\sqrt{2\,(2\pi)^9\,\omega_{\mathbf{p},\pi}}}\;, \tag{4.101}$$

the result in (4.100) is simplified as

$$\Big[\,\hat{\Phi}(\mathbf{x},\theta,t),\hat{\Pi}(\mathbf{x}',\theta^{'},t)\,\Big]_{\star}\;=\;i\,\delta^3\,(\mathbf{x}-\mathbf{x}')\,\delta^6\,(\theta-\theta')-\int\frac{d^9P\,d^9P'}{(2\pi)^9}\,\sqrt{\frac{\omega_{\mathbf{p}',\pi'}}{\omega_{\mathbf{p},\pi}}}\,\sin\left(\frac{p\wedge p'}{2}\right)\,\times$$
$$\times[\,\hat{a}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}\,+\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,\hat{a}_{\mathbf{p},\pi}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}$$
$$-\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}\,+\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}]\,, \tag{4.102}$$

which is the opposite direction followed in [85] where the delta functions are assumed to have the form obtained before. For end, we have the commutation relation involving the momentum operators

$$\Big[\,\hat{\Pi}(\mathbf{x},\theta,t),\hat{\Pi}(\mathbf{x}',\theta^{'},t)\,\Big]_{\star}\;=\int d^9P\int d^9P'\,N_{\mathbf{p},\pi}\,N_{\mathbf{p}',\pi'}\;(-i\omega_{\mathbf{p},\pi})\;(-i\omega_{\mathbf{p}',\pi'})\;\times$$
$$\times\;(-2i)\,\sin\left(\frac{p\wedge p'}{2}\right)[\,\hat{a}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}\,+$$
$$+\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,\hat{a}_{\mathbf{p},\pi}\,e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}$$
$$-\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t-\mathbf{p}'\cdot\mathbf{x}'+\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta')}$$
$$+\,\hat{a}^{\dagger}_{\mathbf{p},\pi}\,\hat{a}^{\dagger}_{\mathbf{p}',\pi'}\,e^{-i(\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p},\pi}\,t+\mathbf{p}'\cdot\mathbf{x}'-\omega_{\mathbf{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta')}\,]\,. \tag{4.103}$$

Notice that what we have done here was to demonstrate the canonical commutation relations using the field operators constructed with DFR phase-space definitions. These canonical relations involving the Moyal-product do not close to the usual case, in which we have obtained combinations between creation and annihilation operators. It is clear that in the commutative limit, these terms involving $\hat{a}$ and $\hat{a}^{\dagger}$ go to zero naturally. If we use the vacuum properties of the operators $\hat{a}$ and $\hat{a}^{\dagger}$, that is, we define a vacuum state $|0\rangle$, such that $\hat{a}_{p,\pi}|0\rangle=0$, so the expected value of the previous commutators in the vacuum state are given by

$$\begin{aligned}
\langle 0|\Big[\hat{\Phi}(\mathbf{x},\theta,t),\hat{\Phi}(\mathbf{x}',\theta',t)\Big]_\star|0\rangle &= 0\,, \\
\langle 0|\Big[\hat{\Phi}(\mathbf{x},\theta,t),\hat{\Pi}(\mathbf{x}',\theta',t)\Big]_\star|0\rangle &= i\,\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\,\delta^6\left(\theta-\theta'\right)\,, \\
\langle 0|\Big[\hat{\Pi}(\mathbf{x},\theta,t),\hat{\Pi}(\mathbf{x}',\theta',t)\Big]_\star|0\rangle &= 0\,.
\end{aligned} \tag{4.104}$$

We can see that the result in (4.103) corroborates the construction of the operator in Eq. (4.8) with a convenient normalization choice and obeying the commutation operators. We believe that this formalism completes the ones depicted in [75] and [70] since in the first one the existence of a NC six-dimensional phase-space is missing since we have shown that the existence of a momentum is connected to Lorentz invariance. Concerning [85], the path here was different since we have demonstrated that the field operator in a NC space-time can be written in terms of plane waves as

$$u_{\mathbf{p},\pi}(\mathbf{x},\theta)=\frac{e^{-i(\mathbf{p}\cdot\mathbf{x}+\pi\cdot\theta)}}{\sqrt{2\,(2\pi)^9\,\omega_{\mathbf{p},\pi}}}\,, \tag{4.105}$$

when we substitute Eq. (4.105) in the Fourier expansion in Eq. (4.94), and the same can accomplished for $\hat{\Pi}$.

We can apply the quantization to the field energy (4.87), so the quantized Hamiltonian operator is given by

$$\hat{H}=\int d^3\mathbf{x}\,d^6\theta\,\frac{1}{2}\Big[\hat{\Pi}^2(\mathbf{x},\theta,t)+(\nabla\hat{\Phi}(\mathbf{x},\theta,t))^2+(\lambda\nabla_\theta\hat{\Phi}(\mathbf{x},\theta,t))^2+m^2\,\hat{\Phi}^2(\mathbf{x},\theta,t)\Big]\,. \tag{4.106}$$

Using the plane wave expansion of the operators $\hat{\Phi}$ and $\hat{\Pi}$, the quantized energy in terms of the creation and annihilation operators is given by

$$\hat{H}=\int d^3\mathbf{p}\;d^6\pi\;\omega_{\mathbf{p},\pi}\,\left(a^\dagger_{\mathbf{p},\pi}\,a_{\mathbf{p},\pi}+\frac{1}{2}\right)\,, \tag{4.107}$$

so that we can obtain the vacuum energy E_0

$$E_0=\langle 0|\hat{H}|0\rangle=\int d^3\mathbf{p}\;d^6\pi\,\frac{1}{2}\,\omega_{\mathbf{p},\pi}\,. \tag{4.108}$$

We can use the Hamiltonian operator (4.107), and the operators (4.94) and (4.95) to calculate the Hamilton's equations of motion as

$$\dot{\hat{\Phi}}(\mathbf{x},\theta,t)=-i\left[\,\hat{\Phi}(\mathbf{x},\theta,t),\hat{H}\,\right]=\hat{\Pi}(\mathbf{x},\theta,t)\,, \tag{4.109}$$

and

$$\dot{\hat{\Pi}}=-i\left[\,\hat{\Pi}(\mathbf{x},\theta,t),\hat{H}\,\right]=\left(\nabla^2+\lambda^2\nabla^2_\theta-m^2\right)\hat{\Phi}(\mathbf{x},\theta,t)\,, \tag{4.110}$$

where we have used the integration by parts when necessary. Notice that, using Eqs. (4.109) and (4.110), we can construct the NC Klein-Gordon equation

$$\ddot{\hat{\Phi}}(\mathbf{x},\theta,t)=\left(\nabla^2+\lambda^2\nabla^2_\theta-m^2\right)\hat{\Phi}(\mathbf{x},\theta,t)\,, \tag{4.111}$$

which shows clearly a different path from [75] since the author did not consider the existence of a canonical momentum.

5 Soldering formalism in noncommutative spacetime

5.1 The canonical soldering formalism

The basic idea of the soldering procedure is to raise a global Noether symmetry of the self and anti-self dual constituents into a local one, but for an effective composite system, consisting of the dual components and an interference term. The objective in [33] is to systematize the procedure like an algorithm and, consequently, to define the soldered action.

An iterative Noether procedure was adopted to lift the global symmetries to the local ones. Therefore, assume that the symmetries in question are being described by the local actions $S_{\pm}(\phi^{\eta}_{\pm})$, invariant under a global multi-parametric transformation

$$\delta\phi^{\eta}_{\pm} = \alpha^{\eta} \;\;, \tag{5.1}$$

where η represents the tensorial character of the basic fields in the dual actions $S_{\pm}$ and, for notational simplicity, will be dropped from now on. Here the $\pm$ subscript is referring to the opposite/complementary aspects of two models at hand, for instance, ϕ_{+} may refer to a left chiral field and ϕ_{-} to a field with right chirality. As it is well known, we can write,

$$\delta S_{\pm} \;=\; J^{\pm}\,\partial_{\pm}\,\alpha \;\;, \tag{5.2}$$

where $J^{\pm}$ are the Noether currents.

Now, under local transformations these actions will not remain invariant, and Noether counterterms become necessary to reestablish the invariance, along with appropriate auxiliary fields $B^{(N)}$, the so-called soldering fields which have no dynamics where the N superscript is referring to the level of the iteration. This makes a wider range of gauge-fixing conditions available. In this way, the N-action can be written as,

$$S_{\pm}(\phi_{\pm})^{(0)} \rightarrow S_{\pm}(\phi_{\pm})^{(N)} = S_{\pm}(\phi_{\pm})^{(N-1)} - B^{(N)}J^{(N)}_{\pm} \;\;. \tag{5.3}$$

Here $J^{(N)}_{\pm}$ are the $N-$iteration Noether currents. For the self and anti-self dual systems we have in mind that this iterative gauging procedure is (intentionally) constructed not to produce invariant actions for any finite number of steps. However, if after N repetitions, the non-invariant piece ends up being only dependent on the gauging parameters, but not on the original fields, there will exist the possibility of mutual cancellation if both gauged version of self and anti-self dual systems are put together. Then, suppose that after N repetitions we arrive at the following simultaneous conditions,

$$\begin{aligned}\delta S_{\pm}(\phi_{\pm})^{(N)} &\neq 0\\ \delta S_B(\phi_{\pm}) &= 0 \ \ ,\end{aligned} \tag{5.4}$$

with S_B being the so-called soldered action

$$S_B(\phi_{\pm}) = S_{+}^{(N)}(\phi_{+}) + S_{-}^{(N)}(\phi_{-}) + \text{Contact Terms}, \tag{5.5}$$

and the "Contact Terms" being generally quadratic functions of the soldering fields. Then we can immediately identify the (soldering) interference term as,

$$S_{int} = \text{Contact Terms} - \sum_{N} B^{(N)} J_{\pm}^{(N)} \ \ . \tag{5.6}$$

Incidentally, these auxiliary fields $B^{(N)}$ may be eliminated, for instance, through theirs equations of motion, from the resulting effective action, in favor of the physically relevant degrees of freedom. It is important to notice that after the elimination of the soldering fields, the resulting effective action will not depend on either self or anti-self dual fields $\phi_{\pm}$ but only in some collective field, say Φ, defined in terms of the original ones in a (Noether) invariant way

$$S_B(\phi_{\pm}) \to S_{eff}(\Phi) \ \ . \tag{5.7}$$

Analyzing in terms of the classical degrees of freedom, it is obvious that we have now a theory with bigger symmetry groups. Once such effective action has been established, the physical consequences of the soldering are readily obtained by simple inspection.

5.2 The NC extension of Minkowski spacetime

As we have seen before, the commutative spacetime is characterized by the canonical Heisenberg commutation relations

$$\left[\hat{\mathcal{X}}^{\mu}, \hat{\mathcal{X}}^{\nu}\right] = 0, \qquad \left[\hat{\mathcal{X}}^{\mu}, \hat{\mathcal{P}}_{\nu}\right] = i\delta^{\mu}_{\nu}, \qquad \left[\hat{\mathcal{P}}_{\mu}, \hat{\mathcal{P}}_{\nu}\right] = 0 \tag{5.8}$$

where $\mu, \nu = 0, 1, 2, 3$. In order to introduce the k-deformed Minkowski spacetime we have [32]

$$\hat{x}^0 = \hat{\mathcal{X}}^0 - \frac{1}{k}\left[\hat{\mathcal{X}}^i, \hat{\mathcal{P}}_j\right]_{+}, \qquad \hat{x}^i = \hat{\mathcal{X}}^i + A\,\eta^{ij}\hat{\mathcal{P}}_j \exp(\frac{2}{k}\hat{\mathcal{P}}_0) \tag{5.9}$$

where $\left[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2\right]_{+} \equiv \frac{1}{2}(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 + \hat{\mathcal{O}}_2\hat{\mathcal{O}}_1)$, $\eta^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$, $i, j = 1, 2, 3$ and A is an arbitrary constant. The NC parameter k has mass dimension and it is real and positive. The Casimir operator of the k-deformed Poincaré's algebra is

$$\hat{\mathcal{C}}_1 = \left(2k\sinh\frac{\hat{p}_0}{2k}\right)^2 - \hat{p}_i^2, \tag{5.10}$$

and for the momentum operators we have

$$\hat{p}_0 = 2k \sinh^{-1} \frac{\hat{\mathscr{P}}_0}{2k}, \qquad \hat{p}_i = \hat{\mathscr{P}}_i. \tag{5.11}$$

With these last results we can construct our NC phase-space $(\hat{x}^\mu, \hat{p}_\nu)$

$$\left[\hat{x}^0, \hat{x}^j\right] = \frac{i}{k}\hat{x}^j, \qquad \left[\hat{x}^i, \hat{x}^j\right] = 0, \qquad \left[\hat{p}_\mu, \hat{p}_\nu\right] = 0, \qquad \left[\hat{x}^i, \hat{p}_j\right] = i\delta^i_j \tag{5.12}$$

$$\left[\hat{x}^0, \hat{p}_0\right] = i\left(\cosh \frac{\hat{p}_0}{2k}\right)^{-1}, \qquad \left[\hat{x}^0, \hat{p}_i\right] = -\frac{i}{k}\hat{p}_i, \qquad \left[\hat{x}^i, \hat{p}_0\right] = 0 \tag{5.13}$$

which satisfies the Jacobi identity. It is easy to see that when $k \to \infty$ we recover the commutative phase-space in Eq.(5.8).

The Casimir operator described above in Eq.(5.10) can now be written in the standard way

$$\hat{\mathscr{C}}_1 = \hat{\mathscr{P}}_0^2 - \hat{\mathscr{P}}_i^2 \tag{5.14}$$

It's easy to see that this selection coincides with the ones in Eq.(5.8).
In the case that $\hat{p}_\mu$ has standard forms like

$$\hat{p}_0 = -i\frac{\partial}{\partial t}, \qquad \hat{p}_i = -i\frac{\partial}{\partial x^i}, \tag{5.15}$$

so that the operator $\hat{\mathscr{P}}_0$ then reads

$$\hat{\mathscr{P}}_0 = -2ik\left(\sin \frac{1}{2k}\frac{\partial}{\partial t}\right). \tag{5.16}$$

In [32] the authors introduced a proper time τ through the operator

$$\hat{\mathscr{P}}_0 \equiv -i\frac{\partial}{\partial \tau} \tag{5.17}$$

and using Eqs.(5.16) and (5.17) we have that

$$2k\left(\sin \frac{1}{2k}\frac{d}{dt}\right)\tau = 1 \tag{5.18}$$

to which the solution is

$$\tau = t + \sum_{n=0}^{+\infty} c_{-n} \exp(-2kn\pi t) \tag{5.19}$$

where $n \geq 0$, $n \in \mathbb{N}$. The coefficients c_{-n} are arbitrary real constants. This property implies a kind of temporal fuzziness coherent in the k-Minkowski spacetime. Notice that as $k \to \infty$, the proper time turns back to the ordinary time variable.

To construct a NC extension of Minkowski spacetime (τ, x^i)(where the NC feature is inside the proper time), let us define a twisted t-coordinate, such that the metric is

$$\begin{aligned} g_{00} &= \dot{\tau}^2 = \left[1 - 2k\pi \sum_{n=0}^{+\infty} nc_{-n} \exp(-2kn\pi t)\right]^2 \\ g_{11} &= g_{22} = g_{33} = -1. \end{aligned} \tag{5.20}$$

So, we can using Eq.(5.20), construct NC models in the commutative framework. Namely, we construct a Lagrangian theory for NC model in the extended framework of the Minkowski spacetime. We will now perform the soldering formalism treatment of some NC Bosonized Chiral Schwinger models.

5.3 Chiral Schwinger model in terms of chiral bosonization

The Chiral Schwinger model is a 2D (1 spatial dimension + 1 time dimension) Euclidean quantum electrodynamics with a Dirac fermion. This model exhibits a spontaneous symmetry breaking of the U(1) symmetry due to a chiral condensate from a pool of instantons [101]. The photon in this model becomes a massive particle at low temperatures. This model can be solved exactly and it is used as a toy model for other complex theories. The bosonization of this theory can be done in several ways that apparently leads to different bosonized models. But these apparently inequivalent models are related by some gauge transformations [29]. Here we shall not enter into the details of this equivalence and just we will discuss the application of the soldering mechanism in the different forms concerning these chiral models. The Chiral Schwinger model is given by following generating functional

$$\mathcal{Z}(A) = \int D\psi D\bar{\psi} \exp\left\{i \int d^2x \mathcal{L}_F\right\} \tag{5.21}$$

when

$$\begin{aligned}
\mathcal{L}_F &= \bar{\psi}\gamma^\mu[i\partial_\mu + e\sqrt{\pi}A_\mu(1-\gamma_5)]\psi && (5.22)\\
&= i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + \bar{\psi}_L\gamma^\mu(i\partial_\mu + 2e\sqrt{\pi}A_\mu)\psi_L. && (5.23)
\end{aligned}$$

Since the right-handed fermion is decoupled, the integration related to it provides a field-independent constant and it can be absorbed by the normalization factor. The remaining path integral can be computed exactly

$$\begin{aligned}
\mathcal{Z}(A) &= \int D\psi_L D\bar{\psi}_L \exp\left[i\int d^2x\, \bar{\psi}_L\gamma^\mu(i\partial_\mu + 2e\sqrt{\pi}A_\mu)\psi_L\right] && (5.24)\\
&= \exp\left[\frac{ie^2}{2}\int d^2x A_\mu \left\{a\eta^{\mu\nu} - (\partial^\mu + \tilde{\partial}^\mu)\frac{1}{\Box}(\partial^\nu + \tilde{\partial}^\nu)\right\} A_\nu\right]
\end{aligned}$$

where a is the Jackiw-Rajaraman regularization constant which must be $a \geq 1$. Because of the d'ALembertian operator in the denominator, the generating functional (5.24) is nonlocal but by introducing a scalar field $\phi(x)$ it can be rewritten in a local form

$$\mathcal{Z}(A) = \int D\phi \exp\left[i\int d^2x \mathcal{L}_B\right], \tag{5.25}$$

where

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu\phi)^2 + e(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu\phi A_\nu + \frac{1}{2}e^2 a A_\mu A^\mu. \tag{5.26}$$

Both Lagrangians (5.24) and (5.26) are equivalent in the sense that both of them lead to the same generating funcional.

In the fermionic model the dynamics only comes from the left-handed fermion and the right-handed one is absent and can be negleted from the beginning. But in the bosonic counterpart, the field $\phi(x)$ contains both left and right-moving components. So the bosonic model has extra degrees of freedom and it is not suitable for describing the Schwinger model. For this reason we can eliminate the extra degrees of freedom by imposing the following "chiral constraint" [102]

$$\Omega(x) \equiv \pi_\phi - \phi' \approx 0, \tag{5.27}$$

where π_ϕ is the canonical momenta associated to $\phi(x)$. The quantum theory that describes this chiral bosonic model is given by the following generating functional

$$\mathcal{Z}_{ch}[A] = \int D\phi \exp\left[i \int d^2x \mathcal{L}_{ch}\right] \tag{5.28}$$

where

$$\mathcal{L}_{ch} = \dot{\phi}\phi' - (\phi')^2 + 2e\phi'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 a A_\mu A^\mu. \tag{5.29}$$

This Lagrangian is a gauged version of the Floreanini and Jackiw's Lagrangian, $\mathcal{L}_0 = \dot{\phi}\phi' - (\phi')^2$ [103].

5.4 The soldering of the NC bosonized CSM

On the 2D extended Minkowski spacetime (τ, x) the Lagrangian (5.29) takes the following action form

$$\begin{aligned}\hat{\mathcal{S}} &= \int d\tau dx \left[\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial x} - (\frac{\partial\phi}{\partial x})^2 + 2e\frac{\partial\phi}{\partial x}(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2\right. \\ &+ \left.\frac{1}{2}e^2 a\eta^{\mu\nu}A_\mu A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right],\end{aligned} \tag{5.30}$$

where the hat symbol "^" again means that the object lives in the extended Minkowski spacetime.

By the coordinate transformation (5.19) we can rewrite the above action in terms of (t, x) with explicit NCy,

$$\begin{aligned}\hat{\mathcal{S}} &= \int dtdx\sqrt{-g}\left[\frac{1}{\dot{\tau}}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} - (\frac{\partial\phi}{\partial x})^2 + 2e\frac{\partial\phi}{\partial x}(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2\right. \\ &+ \left.\frac{1}{2}e^2 a\eta^{\mu\nu}A_\mu A_\nu + \frac{1}{2}\left(\frac{1}{\dot{\tau}}\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x}\right)^2\right],\end{aligned} \tag{5.31}$$

where $\sqrt{-g}$ is Jacobian of the transformation and also the non-trivial measure of the k-deformed Minkowski spacetime. Note that always $\sqrt{-g} =\mid \dot{\tau} \mid$ but here we only focus on the case $\dot{\tau} > 0$.

Until here we have considered only left chiral Schwinger model but the bosonization process gives us both the left and right chiral bosons which depends on the "chiral constraint" that we have imposed on it. The corresponding Lagrangians for these chiral models in the extended Minkowski spacetime are given by

$$\begin{aligned}\hat{\mathcal{L}}_{+} &= \dot{\phi}\phi' - \sqrt{-g}(\phi')^2 + \sqrt{-g}\{2e\phi'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 a[(A_0)^2 - (A_1)^2]\} \\ &+ \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2 \end{aligned} \tag{5.32}$$

$$\begin{aligned}\hat{\mathcal{L}}_{-} &= -\dot{\rho}\rho' - \sqrt{-g}(\rho')^2 + \sqrt{-g}\{2e\rho'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}e^2 b[(A_0)^2 - (A_1)^2]\} \\ &+ \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2. \end{aligned} \tag{5.33}$$

Note that $+$ and $-$ signs are associated to left and right moving chiral bosons, respectively. These models contain NCy through the proper time τ with the finite NC parameter k. In the limit $k \to +\infty$, $\sqrt{-g} = \dot{\tau} = 1$ these Lagrangians turn back to theirs ordinary forms on the Minkowski spacetime.

Now we are ready to sold these two chiral Lagrangians. To accomplish the task we calculate the variation of Eqs.(5.32) and (5.33) under the following local variations

$$\delta\phi = \eta(x) = \delta\rho. \tag{5.34}$$

In fact we are imposing this local symmetry into these models in order to obtain a gauge invariant Lagrangian. Under this variation we have

$$\delta(\hat{\mathcal{L}}_{+} + \hat{\mathcal{L}}_{-}) = (J_{+} + J_{-})\delta B_1 \tag{5.35}$$

where

$$J_{+} = 2\dot{\phi} - 2\sqrt{-g}\phi' + 2e\sqrt{-g}(A_0 - A_1) \tag{5.36}$$

and

$$J_{-} = -2\dot{\rho} - 2\sqrt{-g}\rho' + 2e\sqrt{-g}(A_0 - A_1) \tag{5.37}$$

and B_1 is an auxiliary field that its variation can be defined as

$$\delta B_1 = \partial_x \eta \quad \text{and} \quad \delta B_2 = \partial_t \eta. \tag{5.38}$$

So we must add a counterterm to both original Lagrangians (5.32) and (5.33) to cover the above extra terms. So

$$\hat{\mathcal{L}}_{+1} = \hat{\mathcal{L}}_{+} - J_{+}B_1 \tag{5.39}$$

$$\hat{\mathcal{L}}_{-1} = \hat{\mathcal{L}}_{-} - J_{-}B_1 \tag{5.40}$$

Now let us check the variation of the above Lagrangians

$$\begin{aligned}
\delta\hat{\mathcal{L}}_{+1} &= -(\delta J_+)B_1 = -(2\dot{\eta} - 2\sqrt{-g}\eta')B_1 \\
&= -2B_1(\delta B_2) + 2\sqrt{-g}B_1(\delta B_1) \qquad (5.41)\\
\delta\hat{\mathcal{L}}_{-1} &= -(\delta J_-)B_1 = (2\dot{\eta} - 2\sqrt{-g}\eta')B_1 \\
&= 2B_1(\delta B_2) + 2\sqrt{-g}B_1(\delta B_1). \qquad (5.42)
\end{aligned}$$

As we can see, it is not zero but the extra terms are independent of original fields. So the iteration will finish in this second step by adding another counterterm.

Finally we can sold these two Lagrangians in order to construct an invariant one.

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- - (J_+ + J_-)B_1 - 2\sqrt{-g}(B_1)^2. \qquad (5.43)$$

We can eliminate the auxiliary field B_1 by its equation of motion

$$\frac{\delta\mathcal{W}}{\delta B_1} = 0 \Longrightarrow -(J_+ + J_-) - 4\sqrt{-g}B_1 = 0 \Rightarrow B_1 = \frac{-1}{4\sqrt{-g}}(J_+ + J_-) \qquad (5.44)$$

By substituting Eq.(5.44) into $\mathcal{W}$ we find

$$\hat{\mathcal{W}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- + \frac{1}{8\sqrt{-g}}(J_+ + J_-)^2. \qquad (5.45)$$

Here we define a new field $\Psi = \phi - \rho$. By this definition we can rewrite $\mathcal{W}$ in a compact and nice form

$$\hat{\mathcal{W}} = -\frac{\sqrt{-g}}{2}\Psi'^2 + \frac{1}{2\sqrt{-g}}\dot{\Psi}^2 + 2e\dot{\Psi}(A_0 - A_1) + 2\xi \qquad (5.46)$$

where ξ is

$$\xi = \sqrt{-g}\{\frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{4}e^2(a+b)[(A_0)^2 - (A_1)^2]\} + \frac{1}{2\sqrt{-g}}(\dot{A}_1 - \sqrt{-g}A_0')^2 \qquad (5.47)$$

As the final result, the action (5.46) is not "chiral" theory anymore and it has a bigger symmetry group than the two initial models. To this aim, we have soldered the two chiral models and as a consequence we gained an additional term in the final Lagrangian that was absent initially. One of the peculiar futures of this action is that the electromagnetic field interacts just with the temporal derivative of soldered field. This peculiarity has its origin in the noncovariant initial Jackiw-Floreanini Lagrangian. In fact one can decompose the above action into two distinct ones using the dual projection approach. The result is a self-dual and a free massive scalar fields.

This mechanism in some sense is analogous to adding a mass term into the Dirac action. Without this mass term the Dirac equation describes two chiral electrons and by adding the mass, we have merged these two chiral electrons to obtain the real electron.

5.5 The soldering of the generalized bosonized CSM

Bassetto et al. [104] have suggested the generalized chiral Schwinger model (GCSM), i.e., a vector and axial-vector theory characterized by a parameter which interpolates between pure vector and chiral Schwinger models. This 2D model is given by the action

$$\hat{\mathcal{S}} = \int \mathrm{d}t\mathrm{d}x \left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) + eA_\mu\left(\epsilon^{\mu\nu} - r\eta^{\mu\nu}\right)\partial_\nu\phi + \frac{1}{2}e^2 aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{5.48}$$

The quantity r is a real interpolating parameter between the vector ($r = 0$) and the chiral Schwinger models ($r = \pm 1$). This action can be rewritten in the extended Minkowski spacetime

$$\hat{\mathcal{L}} = \frac{1}{2\sqrt{-g}}\dot{\phi}^2 - \frac{\sqrt{-g}}{2}\phi'^2 - k_1\dot{\phi} + k_2\phi' + \xi \tag{5.49}$$

where

$$\begin{aligned}
k_1 &= e(rA_0 + A_1) && (5.50)\\
k_2 &= e\sqrt{-g}(A_0 + rA_1) && (5.51)\\
\xi &= \frac{1}{2\sqrt{-g}}\left(\dot{A}_1 - \sqrt{-g}A_0'\right)^2 + \frac{1}{2}e^2 a\sqrt{-g}\left[(A_0)^2 - (A_1)^2\right]. && (5.52)
\end{aligned}$$

By defining the value of the parameter r in two extreme points ± 1 we obtain two chiral Lagrangians

$$\begin{aligned}
\hat{\mathcal{L}}_+ &= \frac{1}{2\sqrt{-g}}\dot{\phi}^2 - \frac{\sqrt{-g}}{2}\phi'^2 - e(A_0 + A_1)\dot{\phi} + e\sqrt{-g}(A_0 + A_1)\phi' \\
&+ \frac{1}{2\sqrt{-g}}\left(\dot{A}_1 - \sqrt{-g}A_0'\right)^2 + \frac{1}{2}ae^2\sqrt{-g}\left[(A_0)^2 - (A_1)^2\right] && (5.53)\\
\hat{\mathcal{L}}_- &= \frac{1}{2\sqrt{-g}}\dot{\rho}^2 - \frac{\sqrt{-g}}{2}\rho'^2 - e(-A_0 + A_1)\dot{\rho} + e\sqrt{-g}(A_0 - A_1)\rho' \\
&+ \frac{1}{2\sqrt{-g}}\left(\dot{A}_1 - \sqrt{-g}A_0'\right)^2 + \frac{1}{2}be^2\sqrt{-g}\left[(A_0)^2 - (A_1)^2\right] && (5.54)
\end{aligned}$$

where a and b are the Jackiw-Rajaraman coefficients for each chirality, respectively. Here, by means of iterative Noether embedding procedure, we will transfom both Lagrangians (5.53) and (5.54) into two embedded Lagrangians which are invariant under transformations $\delta\phi = \eta(x)$ and $\delta\rho = \eta(x)$. After that we will be able to sold these new Lagrangians in order to yield an invariant one that describes a fermionic system. By varying the Lagrangians with respect to the variables $\partial_t\Phi$ and $\partial_x\Phi$, ($\Phi = (\phi, \rho)$), we obtain the following Noether currents

$$\begin{aligned}
J_{1+} &= \frac{1}{\sqrt{-g}}\dot{\phi} - e(A_0 + A_1) && (5.55)\\
J_{2+} &= -\sqrt{-g}\left[\phi' - e(A_0 + A_1)\right] && (5.56)\\
J_{1-} &= \frac{1}{\sqrt{-g}}\dot{\rho} + e(A_0 - A_1) && (5.57)\\
J_{2-} &= -\sqrt{-g}\left[\rho' - e(A_0 - A_1)\right]. && (5.58)
\end{aligned}$$

After two soldering iteration steps and adding the counterterms to the original Lagrangians we find

$$\hat{\mathcal{L}}_{+}^{(2)} = \mathcal{L}_{+} - J_{1+}B_1 - J_{2+}B_2 + \tfrac{1}{2\sqrt{-g}}(B_1)^2 - \tfrac{\sqrt{-g}}{2}(B_2)^2 + \xi_{+} \tag{5.59}$$

$$\hat{\mathcal{L}}_{-}^{(2)} = \mathcal{L}_{-} - J_{1-}B_1 - J_{2-}B_2 + \tfrac{1}{2\sqrt{-g}}(B_1)^2 - \tfrac{\sqrt{-g}}{2}(B_2)^2 + \xi_{-} \tag{5.60}$$

where $\xi_{\pm}$ are non-dynamical terms of $\mathcal{L}_{\pm}$. The embedding process ends after these two steps and these Lagrangians are invariant under the desired transformation $\delta\phi = \eta(x)\delta\rho$. Now we can solder them by adding up two Lagrangians Eqs.(5.59) and (5.60)

$$\begin{aligned} \hat{\mathcal{W}} &= \hat{\mathcal{L}}_{+}^{(2)} + \hat{\mathcal{L}}_{-}^{(2)} \qquad (5.61) \\ &= \hat{\mathcal{L}}_{+} + \hat{\mathcal{L}}_{-} - (J_{1+} + J_{1-})B_1 - (J_{2+} + J_{2-})B_2 + \frac{1}{\sqrt{-g}}(B_1)^2 - \sqrt{-g}(B_2)^2 \end{aligned}$$

To express this Lagrangian just in terms of the original fields, we can eliminate B_1 and B_2 easily by using their equation of motions, it reads:

$$B_1 = \tfrac{\sqrt{-g}}{2}(J_{1+} + J_{1-}) \tag{5.62}$$

$$B_2 = -\tfrac{1}{2\sqrt{-g}}(J_{2+} + J_{2-}). \tag{5.63}$$

After substituting these results into $\mathcal{W}$, defining a new field $\Psi = \phi - \rho$ and set the Jackiw-Rajaraman coefficients $a = 1 = b$, for simplicity, we can write that

$$\begin{aligned} \hat{\mathcal{W}} &= \frac{1}{4\sqrt{-g}}\dot{\Psi}^2 - \frac{\sqrt{-g}}{4}\Psi'^2 - eA_0\dot{\Psi} + eA_1\sqrt{-g}\Psi' \\ &+ \frac{1}{2\sqrt{-g}}\left(\dot{A}_1 - \sqrt{-g}A_0'\right)^2 + e^2\sqrt{-g}\left[(A_0)^2 - (A_1)^2\right]. \end{aligned} \tag{5.64}$$

This Lagrangian describes a 2D fermionic system and has a larger symmetry group than the initial Lagrangians (5.53) and (5.54). As the previous case, the soldering process included an extra noton term into the original Lagrangians to fuse the chiral states. This non-dynamical term acquires dynamics upon quantization [33].

5.6 The soldering of the gauge invariant generalized bosonized CSM

In [105], the authors have introduced the Wess-Zumino (WZ) term for the GCSM and constructed its gauge invariant formulation by adding the WZ term into the Lagrangian of the model. This gauge invariant model is described by

$$\begin{aligned} \hat{\mathcal{S}} &= \int \mathrm{d}t\mathrm{d}x \left\{\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) + eA^\mu\left(\epsilon_{\mu\nu} - r\eta_{\mu\nu}\right)\partial^\nu\phi + \frac{1}{2}e^2aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right. \\ &\left. +\frac{1}{2}\left(a - r^2\right)\left(\partial_\mu\theta\right)\left(\partial^\mu\theta\right) + eA^\mu\left[r\epsilon_{\mu\nu} + \left(a - r^2\right)\eta_{\mu\nu}\right]\partial^\nu\theta\right\}, \end{aligned} \tag{5.65}$$

where $\theta(x)$ is the WZ field. The Lagrangians of left/right moving bosons are given by defining the parameter r at its two extreme points ± 1

$$\begin{aligned}
\hat{\mathcal{L}}_{+} &= \frac{1}{2\sqrt{-g}}(\dot{\phi})^2 - \frac{\sqrt{-g}}{2}(\phi')^2 - b_1\dot{\phi} + b_1\sqrt{-g}\phi' \qquad (5.66)\\
&+ \frac{b_2}{\sqrt{-g}}(\dot{\theta})^2 - b_2\sqrt{-g}(\theta')^2 + b_3\dot{\theta} + b_4\theta' + \xi_{+}\\
\hat{\mathcal{L}}_{-} &= \frac{1}{2\sqrt{-g}}(\dot{\rho})^2 - \frac{\sqrt{-g}}{2}(\rho')^2 - b_5\dot{\rho} + b_5\sqrt{-g}\rho' \qquad (5.67)\\
&+ \frac{b_2'}{\sqrt{-g}}(\dot{\eta})^2 - b_2'\sqrt{-g}(\eta')^2 + b_6\dot{\eta} + b_7\eta' + \xi_{-}
\end{aligned}$$

where η is also another WZ field and

$$\begin{aligned}
b_1 &\equiv e(A_0 + A_1), \qquad b_2 \equiv \frac{a-1}{2}, \qquad b_2' \equiv \frac{b-1}{2}, \qquad b_3 \equiv e\left[A_0(a-1) - A_1\right]\\
b_4 &\equiv e\sqrt{-g}\left[A_0 - A_1(a-1)\right], \qquad b_5 \equiv e(A_0 - A_1),\\
b_6 &\equiv e\left[A_0(b-1) + A_1\right], \qquad b_7 \equiv e\sqrt{-g}\left[-A_0 - A_1(b-1)\right]\\
\xi_{\pm} &\equiv \frac{1}{2\sqrt{-g}}(\dot{A}_1)^2 - \frac{\sqrt{-g}}{2}(A_0')^2 + \frac{\sqrt{-g}}{2}e^2\binom{a}{b}\left[(A_0)^2 - (A_1)^2\right] - \dot{A}_1 A_0'. \qquad (5.68)
\end{aligned}$$

The goal is gauging these Lagrangians under the following transformations

$$\begin{aligned}
\delta\phi &= \delta\rho = \alpha(x)\\
\delta\theta &= \delta\eta = \beta(x). \qquad (5.69)
\end{aligned}$$

The Noether currents under these transformations are

$$\begin{aligned}
J_{1+} &= \frac{1}{\sqrt{-g}}\dot{\phi} - b_1, \qquad & J_{1-} &= \frac{1}{\sqrt{-g}}\dot{\rho} - b_5,\\
J_{2+} &= -\sqrt{-g}\phi' + b_1\sqrt{-g}, \qquad & J_{2-} &= -\sqrt{-g}\rho' + b_5\sqrt{-g},\\
J_{3+} &= \frac{2b_2}{\sqrt{-g}}\dot{\theta} + b_3, \qquad & J_{3-} &= \frac{2b_2'}{\sqrt{-g}}\dot{\eta} + b_6,\\
J_{4+} &= -2b_2\sqrt{-g}\theta' + b_4, \qquad & J_{4-} &= -2b_2'\sqrt{-g}\eta' + b_7. \qquad (5.70)
\end{aligned}$$

The first iteration Lagrangians read

$$\begin{aligned}
\hat{\mathcal{L}}_{+}^{(1)} &= \hat{\mathcal{L}}_{+} - J_{1+}B_1 - J_{2+}B_2 - J_{3+}B_3 - J_{4+}B_4\\
\hat{\mathcal{L}}_{-}^{(1)} &= \hat{\mathcal{L}}_{-} - J_{1-}B_1 - J_{2-}B_2 - J_{3-}B_3 - J_{4-}B_4 \qquad (5.71)
\end{aligned}$$

where B_1, B_2, B_3 and B_4 are new auxiliaries fields which have the following variations

$$\delta B_1 = \partial_t\alpha, \qquad \delta B_2 = \partial_x\alpha, \qquad \delta B_3 = \partial_t\beta, \qquad \delta B_4 = \partial_t\beta. \qquad (5.72)$$

The variation of the first iterated Lagrangians are given by

$$\delta\hat{\mathcal{L}}_{+}^{(1)} = -\frac{1}{\sqrt{-g}}(\delta B_1)B_1 + \sqrt{-g}(\delta B_2)B_2 - \frac{2b_2}{\sqrt{-g}}(\delta B_3)B_3 + 2b_2\sqrt{-g}(\delta B_4)B_4 \qquad (5.73)$$

$$\delta\hat{\mathcal{L}}_{-}^{(1)} = -\frac{1}{\sqrt{-g}}(\delta B_1)B_1 + \sqrt{-g}(\delta B_2)B_2 - \frac{2b_2}{\sqrt{-g}}(\delta B_3)B_3 + 2b_2\sqrt{-g}(\delta B_4)B_4. \qquad (5.74)$$

As it can be seen, these variations are completely independent of original fields, therefore the embedding process finished here and by adding the counterterms associated to these variations we can obtain our desired invariant Lagrangian. Now we are ready to fuse both Lagrangians (5.71) by adding them up and introducing a counterterm

$$\begin{aligned} \hat{\mathscr{W}} &= \hat{\mathscr{L}}_+ + \hat{\mathscr{L}}_- - J_{1+}B_1 - J_{2+}B_2 - J_{3+}B_3 - J_{4+}B_4 - J_{1-}B_1 - J_{2-}B_2 - J_{3-}B_3 - J_{4-}B_4 \\ &+ \frac{1}{\sqrt{-g}}(B_1)^2 - \sqrt{-g}(B_2)^2 + \frac{2b_2}{\sqrt{-g}}(B_3)^2 - 2b_2\sqrt{-g}(B_4)^2 \end{aligned} \tag{5.75}$$

where we have fixed the Jackiw-Rajaraman coefficients $a = b$ for simplicity. To express the final result only in terms of the original fields, one can eliminate the auxiliary fields by using their equations of motions

$$\begin{aligned} B_1 &= \frac{\sqrt{-g}}{2}(J_{1+} + J_{1-}) \\ B_2 &= \frac{-1}{2\sqrt{-g}}(J_{2+} + J_{2-}) \\ B_3 &= \frac{\sqrt{-g}}{4b_2}(J_{3+} + J_{3-}) \\ B_4 &= \frac{-1}{4b_2\sqrt{-g}}(J_{4+} + J_{4-}). \end{aligned} \tag{5.76}$$

By substituting these results into Eq.(5.75) and introducing two soldering fields $\Psi = \phi - \rho$ and $\Omega = \theta - \eta$ we obtain an effective action

$$\begin{aligned} \hat{\mathscr{W}}_{eff} &= \frac{1}{4\sqrt{-g}}(\dot{\Psi})^2 - \frac{\sqrt{-g}}{4}(\Psi')^2 - eA_0\dot{\Psi} + e\sqrt{-g}A_1\Psi' + \frac{b_2}{2\sqrt{-g}}(\dot{\Omega})^2 \\ &- \frac{b_2\sqrt{-g}}{2}(\Omega')^2 - eA_1\dot{\Omega} + \frac{1}{2}\left[eA_0 + e\sqrt{-g}(A_0 - 2A_1b_2) + 2eA_1b_2\right]\Omega' \\ &- 2e^2b_2\sqrt{-g}(A_0)^2 + \frac{e^2\sqrt{-g}}{8b_2}(A_0 - 2A_1b_2)^2 - \frac{2e^2A_0}{8b_2}(A_0 - 2A_1b_2) \\ &+ \frac{e^2}{8\sqrt{-g}b_2}(A_0)^2 + \frac{e^2}{2\sqrt{-g}}A_0A_1 + \frac{e^2b_2}{2\sqrt{-g}}(A_1)^2 - \frac{e^2}{2}A_1(A_0 - 2A_1b_2) + 2\xi \end{aligned} \tag{5.77}$$

where $\xi = \xi_- + \xi_+$. The initial Lagrangians were invariant under a semilocal gauge group, but this effective Lagrangian is invariant under the local version of the initial gauge group and moreover it is invariant under gauge transformations (5.69).

Maybe someone asks about the counterpart of this model in the commutative spacetime. We can find it just by putting $\sqrt{-g} = 1$. It reads

$$\begin{aligned} \mathscr{W}_{eff} &= \frac{1}{4}\partial_\mu\Psi\partial^\mu\Psi + e\epsilon^{\mu\nu}A_\mu\partial_\nu\Psi + (a-1)\,\partial_\mu\Omega\partial^\mu\Omega - eA_\mu\epsilon^{\mu\nu}\partial_\nu\Omega \\ &+ \frac{1}{2}e^2aA_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{5.78}$$

where $\xi' = \xi|_{\sqrt{-g}=1}$. We have succeeded in including the effects of interference between rightons and leftons (right/left moving scalar). Consequently, these components have lost their individuality in favor of a new, gauge invariant, collective field that does not depend on ϕ or ρ separately.

As it can be seen, this Lagrangian is apparently different from the initial ones and the new fields Ψ and Ω are not chiral anymore. If we fix the Jackiw-Rajaraman coefficients $a = b = 1$ the field Ω becomes non-dynamical and will just interact with electromagnetic field. The combination of the massless modes led to a massive vectorial mode as a consequence of the chiral interference. The noton field, that is defined before, propagates neither to the left nor to the right.

5.7 The massive Thirring model

The Thirring model is an exactly solvable QFT that describes the self-interactions of a Dirac theory in (2+1) dimensions. For the first time S. Coleman discovered an equivalence between this model and Sine-Gordon model which is bosonic theory [106]. In order to study another example of the soldering formalism we will consider the bosonization of Thirring model and after that we will show how one can fuse the bosonized version of this model. Here we briefly review the bosonization process of this model.

The generating functional of the massive Thirring model in Minkowski spacetime is given by

$$\mathcal{Z}(k) = \int D\psi D\bar{\psi} \exp\left\{i \int d^3x \left[\bar{\psi}(i\partial\!\!\!/ + m)\psi - \frac{\lambda^2}{2} j_\mu j^\mu + \lambda j_\mu k^\mu\right]\right\} \tag{5.79}$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$ is a fermionic current. The fermionic current can be eliminated by using an auxiliary variable

$$\mathcal{Z}(k) = \int D\psi D\bar{\psi} Df_\mu \exp\left\{i \int d^3x \left[\bar{\psi}(i\partial\!\!\!/ + m + \lambda(f\!\!\!/ + k\!\!\!/))\psi + \frac{1}{2} f_\mu f^\mu\right]\right\}. \tag{5.80}$$

In 3-dimensional spacetime this integration can not be done exactly. Under certain limiting conditions, however, this integration leads to an exact expression. A particularly effective choice is the large mass limit in which case the fermion determinant yields a local form. Incidentally, any other value of the mass leads to a nonlocal structure [107]. The large mass limit is therefore very special. The leading term in this limit was calculated by various ways [108] and it can be shown to yield the Chern-Simons three form. Thus the generating functional for the massive Thirring model in the large mass limit is given by

$$Z[k] = \int Df_\mu \exp\left(i \int d^3x \left(\frac{\lambda^2}{8\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda + \frac{1}{2} f_\mu f^\mu + \frac{\lambda^2}{4\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\sigma} k^\mu \partial^\nu f^\sigma\right)\right), \tag{5.81}$$

where the signature of the topological terms is dictated by the corresponding signature of the fermionic mass term. In obtaining the above result a local counter term has been ignored. Such terms manifest the ambiguity in defining the time ordered product to compute the correlation functions [109]. The Lagrangian in the above partition function defines a self dual model introduced earlier [110]. The massive Thirring model, in the relevant limit, therefore can be bosonized to a self dual model. It is useful to clarify the meaning of this self duality. The equation of motion in the absence of sources is given by

$$f_\mu = -\frac{\lambda^2}{4\pi}\frac{m}{\mid m \mid}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda \tag{5.82}$$

from which the following relations may be easily verified

$$\begin{aligned} \partial_\mu f^\mu &= 0 \\ \left(\Box + M^2\right) f_\mu &= 0 \quad ; \; M = \frac{4\pi}{\lambda^2}. \end{aligned} \tag{5.83}$$

A field dual to f_μ is defined as

$$\tilde{f}_\mu = \frac{1}{M}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda \tag{5.84}$$

where the mass parameter M is inserted for dimensional reasons. Repeating the dual operation, we find that

$$\tilde{\tilde{f}}_\mu = \frac{1}{M}\epsilon_{\mu\nu\lambda}\partial^\nu \tilde{f}^\lambda = f_\mu \tag{5.85}$$

was obtained by exploiting (5.83), thereby validating the definition of the dual field. Combining these results with (5.82), we conclude that

$$f_\mu = -\frac{m}{\mid m \mid}\tilde{f}_\mu. \tag{5.86}$$

Hence, depending on the sign of the fermion mass term, the bosonic theory corresponds to a self-dual or an anti self-dual model. Likewise, the Thirring current leads to the topological current

$$j_\mu = \frac{\lambda}{4\pi}\frac{m}{\mid m \mid}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho. \tag{5.87}$$

The close connection to the two dimensional analysis is now clear. There the starting point was to consider two distinct fermionic theories with opposite chiralities. The analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures,

$$\begin{aligned} \mathcal{L}_+ &= \bar{\psi}\left(i\partial\!\!\!/ + m\right)\psi - \frac{\lambda^2}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2 \\ \mathcal{L}_- &= \bar{\xi}\left(i\partial\!\!\!/ - m'\right)\xi - \frac{\lambda^2}{2}\left(\bar{\xi}\gamma_\mu\xi\right)^2. \end{aligned} \tag{5.88}$$

Then the bosonized Lagrangians are, respectively,

$$\begin{aligned} \mathcal{L}_+ &= \frac{1}{2M}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda + \frac{1}{2}f_\mu f^\mu \\ \mathcal{L}_- &= -\frac{1}{2M}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda + \frac{1}{2}g_\mu g^\mu, \end{aligned} \tag{5.89}$$

where f_μ and g_μ are the distinct bosonic vector fields. The current bosonization formula in both cases are given by

$$
\begin{aligned}
j^+_\mu &= \bar{\psi}\gamma_\mu\psi = \frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho \\
j^-_\mu &= \bar{\xi}\gamma_\mu\xi = -\frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu g^\rho .
\end{aligned}
\tag{5.90}
$$

These models are known as self and anti-self dual models in the literature.

5.8 The soldering of NC (anti)self-dual models

On the extended Minkowski spacetime (τ, x) the Lagrangian (5.89) takes the following action form

$$
\hat{\mathcal{S}}_\pm = \int d\tau d^2x \left[\frac{1}{2}h^\mu h_\mu \pm \frac{1}{2M}\left(\epsilon_{\mu 0\lambda}h^\mu \frac{\partial h^\lambda}{\partial \tau} + \epsilon_{\mu i\lambda}h^\mu \partial^i h^\lambda\right)\right] \tag{5.91}
$$

where $h^\mu = f^\mu, g^\mu$.

After making the coordinate transformation, we obtain the action written in terms of the coordinates (t, x),

$$
\hat{\mathcal{S}}_\pm = \int dtd^2x\sqrt{-g}\Big[\frac{1}{2}h^\mu h_\mu \pm \frac{1}{2M}\epsilon_{\mu i\lambda}h^\mu\partial^i h^\lambda\Big] \pm \frac{1}{2M}\epsilon_{\mu 0\lambda}h^\mu\frac{\partial h^\lambda}{\partial t}. \tag{5.92}
$$

Taking a hint from the two dimensional case, let us consider the gauging of the following symmetry

$$
\delta f_\mu = \delta g_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho\alpha^\sigma . \tag{5.93}
$$

Under these transformations the bosonized Lagrangians change as

$$
\delta\hat{\mathcal{S}}_\pm = \int dtd^2x\left[\sqrt{-g}\left\{\epsilon^{\mu\rho\sigma}h_\mu \pm \frac{1}{M}\epsilon_{\mu i\lambda}\epsilon^{\mu\rho\sigma}\partial^i h^\lambda\right\} \pm \frac{1}{M}\epsilon_{\mu 0\lambda}\epsilon^{\mu\rho\sigma}\partial^0 h^\lambda\right]\partial_\rho\alpha_\sigma . \tag{5.94}
$$

We can identify the Noether currents

$$
J^{\rho\sigma}_\pm(h_\mu) = \sqrt{-g}\left\{\epsilon^{\mu\rho\sigma}h_\mu \pm \frac{1}{M}\epsilon_{\mu i\lambda}\epsilon^{\mu\rho\sigma}\partial^i h^\lambda\right\} \pm \frac{1}{M}\epsilon_{\mu 0\lambda}\epsilon^{\mu\rho\sigma}\partial^0 h^\lambda . \tag{5.95}
$$

As a comment about the form of the variation (5.93) we can say that the more simpler form such as the one we have assumed in 2D case, will not be suitable and the variations cannot be combined to give a single structure like (5.95). Now we introduce the auxiliary field coupled with the antisymmetric currents. In the two dimensional case this field was a vector. In the three dimensional case, as a natural generalization, we adopt an antisymmetric second rank Kalb-Ramond tensor field $B_{\rho\sigma}$ where its transformation is given by

$$
\delta B_{\rho\sigma} = \partial_\rho\alpha_\sigma - \partial_\sigma\alpha_\rho \tag{5.96}
$$

It is worthwhile to mention that in the canonical NC approach one must include the variation of current associated with the NC field/parameter to the transformation of the auxiliary tensor field to obtain an effective Lagrangian after soldering [111].
To eliminate the non-vanishing change (5.94), we add a counter term to the original Lagrangian. So the first iterated Lagrangians are

$$\mathcal{L}_{\pm}^{(1)} = \mathcal{L}_{\pm} - \frac{1}{2} J_{\pm}^{\rho\sigma}(h_\mu) B_{\rho\sigma} \tag{5.97}$$

which transform as,

$$\delta\mathcal{L}_{\pm}^{(1)} = -\frac{1}{2}\delta J_{\pm}^{\rho\sigma} B_{\rho\sigma}. \tag{5.98}$$

The variation of the currents coupled with the auxiliary field is

$$\delta J_{\pm}^{\rho\sigma} B_{\rho\sigma} = \sqrt{-g}\Big[\delta B^{\rho\sigma} B_{\rho\sigma} \mp \frac{1}{M}\epsilon^{\lambda\gamma\theta}(\partial^i\partial_\gamma\alpha_\theta)B_{i\lambda}\Big] \mp \frac{2}{M}\epsilon^{\lambda\gamma\theta}(\partial^0\partial_\gamma\alpha_\theta)B_{0\lambda}. \tag{5.99}$$

As we can see, the above Lagrangians also are not invariant under the transformations (5.93), hence we must go further and add another counter term. As a key point, in the soldering formalism the invariance of one Lagrangian alone is not desired. We are looking for a combination of both Lagrangians that be gauge invariant. To this aim, the second iteration Lagrangians is defined as

$$\mathcal{L}_{\pm}^{(2)} = \mathcal{L}_{\pm}^{(1)} + \frac{\sqrt{-g}}{4} B^{\rho\sigma} B_{\rho\sigma}. \tag{5.100}$$

By this definition a straightforward algebra shows that the following combination is invariant under transformation (5.93) and (5.96),

$$\begin{aligned}\mathcal{L}_S &= \mathcal{L}_{+}^{(2)} + \mathcal{L}_{-}^{(2)} \\ &= \mathcal{L}_{+} + \mathcal{L}_{-} - \frac{1}{2} B^{\rho\sigma}\left(J_{\rho\sigma}^{+}(f) + J_{\rho\sigma}^{-}(g)\right) + \frac{\sqrt{-g}}{2} B^{\rho\sigma} B_{\rho\sigma}.\end{aligned} \tag{5.101}$$

The gauging of the symmetry is therefore complete now. But the final result would be more interesting if we express the above Lagrangian in terms of the original fields. By using the equation of motion of field $B_{\rho\sigma}$ we can eliminate this auxiliary field

$$B_{\rho\sigma} = \frac{1}{2\sqrt{-g}}\left(J_{\rho\sigma}^{+}(f) + J_{\rho\sigma}^{-}(g)\right). \tag{5.102}$$

Including this solution into (5.101) the final soldered Lagrangian is expressed only in terms of the original fields,

$$\mathcal{L}_S = \mathcal{L}_{+} + \mathcal{L}_{-} - \frac{1}{8\sqrt{-g}}\left(J_{\rho\sigma}^{+}(f) + J_{\rho\sigma}^{-}(g)\right)\left(J^{+\rho\sigma}(f) + J^{-\rho\sigma}(g)\right). \tag{5.103}$$

The crucial point of soldering formalism comes now, by using the explicit structures for the currents, the above Lagrangian is no longer a function of f_μ and g_μ separately, but solely on the combination

$$A_\mu = \frac{1}{\sqrt{2}M}(f_\mu - g_\mu). \tag{5.104}$$

By this field redefinition we obtain the final effective action as

$$\mathcal{L}_S = \frac{M^2\sqrt{-g}}{2}A^\mu A_\mu + \partial_i A_0 \partial^0 A^i - \frac{1}{2\sqrt{-g}}\partial_0 A_i \partial^0 A^i - \frac{\sqrt{-g}}{2}(\partial^i A^0 \partial_i A_0 + \partial_i A_j \partial^i A^j - \partial_j A_i \partial^i A^j). \tag{5.105}$$

In the usual commutative Minkowski spacetime we yield the Proca theory by soldering two (anti)self-dual theories [109]. As a generalization we claim the Lagrangian (5.105) to be the NC version of the Abelian Proca theory in the κ-deformed (2+1)D Minkowski spacetime. In order to check that our calculation is correct we can obtain directly this Lagrangian by applying the coordinate transformation $(\tau, x) \to (t, x)$ in Proca theory. The Abelian Proca model on the extended Minkowski spacetime (τ, x) is

$$\begin{aligned}
\hat{\mathcal{S}} &= \int d\tau d^2x \Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}A^\mu A_\mu \Big] \\
&= \int d\tau d^2x \Big(-\frac{1}{2}\Big[\frac{\partial A^i}{\partial \tau}(\frac{\partial A_i}{\partial \tau} - \frac{\partial A_0}{\partial x^i}) + \frac{\partial A^0}{\partial x_i}(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial \tau}) \\
&+ \frac{\partial A^j}{\partial x_i}(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j})\Big] + \frac{M^2}{2}A^\mu A_\mu \Big)
\end{aligned} \tag{5.106}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. By coordinate transformation (5.19) we can rewrite the above actions in term of (t, x) with explicit NCy,

$$\begin{aligned}
\hat{\mathcal{S}} &= \int dt d^2x \sqrt{-g}\Big(-\frac{1}{2}\Big[\frac{1}{\sqrt{-g}}\frac{\partial A^i}{\partial t}(\frac{1}{\sqrt{-g}}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i}) + \frac{\partial A^0}{\partial x_i}(\frac{\partial A_0}{\partial x^i} - \frac{1}{\sqrt{-g}}\frac{\partial A_i}{\partial t}) \\
&+ \frac{\partial A^j}{\partial x_i}(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j})\Big] + \frac{M^2}{2}A^\mu A_\mu \Big).
\end{aligned} \tag{5.107}$$

Here we have assumed that $\dot{\tau} = \sqrt{-g} > 0$. After some straightforward manipulation we find

$$\begin{aligned}
\hat{\mathcal{S}} &= \frac{1}{2}\int dt d^2x \Big(2\partial^0 A^i \partial_i A_0 + \sqrt{-g}\partial^i A^j \partial_j A_i - \sqrt{-g}\partial^i A^j \partial_i A_j \\
&- \sqrt{-g}\partial^i A^0 \partial_i A_0 - \frac{1}{\sqrt{-g}}\partial^0 A^i \partial_0 A_i + M^2\sqrt{-g}A^\mu A_\mu \Big).
\end{aligned} \tag{5.108}$$

As we expected this action coincide with the model described by Lagrangian (5.105).

An interesting observation about this NC version is that besides the modification of the field dynamics in this new spacetime, the mass term is also changed and it is not equal to the usual Minkowski spacetime so, the particle associated to this field must have a different mass in this spacetime.

It is noteworthy that the transformations (5.93) are not the unique ones that lead to this result. We also can take the transformation

$$\delta f_\mu = -\delta g_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho \alpha^\sigma. \tag{5.109}$$

By assuming the above transformation and defining the final soldered field

$$A_\mu = \frac{1}{\sqrt{2}M}(f_\mu - g_\mu) \tag{5.110}$$

we arrive at the same Lagrangian as (5.105). This led the authors of paper [112] to the idea of generalizing the soldering formalism. As it was mentioned before, the basic idea of soldering was that adding two independent dual Lagrangians does not give us new information and for obtaining a gauge invariant model we have to fuse two Lagrangians via the Noether procedure. This idea was successfully applied to different models in various dimensions such as chiral Schwinger model with opposite chiralities. Some years after proposing this idea it was shown that the usual sum of opposite chiral bosons models is, in fact, gauge invariant and corresponds to a composite model, where the component models are the vector and axial Schwinger models [112]. As a consequence, we can reinterpret the soldering formalism as a kind of degree of freedom reduction mechanism.

In the case at hand, two transformations (5.93) and (5.109) result in the same effective action but in a general case we may obtain two apparently different actions. For example, if we add an interaction term to the Lagrangians (5.89) the final result will be different. This property is the subject of generalized soldering formalism [112]. Now this question may arise whether these two actions are describing two distinct phenomena. However, by calculating the generating functional of these two Lagrangians we yield the same result. This shows that we are dealing with the same physics but described by different Lagrangians.

6 Noncommutative Jackiw-Pi model

6.1 The Jackiw-Pi Theory

The JP model is a 3D gauge invariant, massive, parity preserving theory governed by the Lagrangian [34, 35]

$$\mathcal{S} = Tr \int d^3x \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} G^{\mu\nu} G_{\mu\nu} - m\epsilon^{\mu\nu\rho} F_{\mu\nu}\phi_\rho \right) \tag{6.1}$$

where A_μ and ϕ_μ are vector bosonic fields and m is a mass parameter. The 2-form curvature $F^{(2)} = dA^{(1)} - i\left(A^{(1)} \wedge A^{(1)}\right) = \frac{1}{2!}\left(dx^\mu \wedge dx^\nu\right) F_{a\mu\nu}\, T^a$ defines the curvature tensor $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i\left[A_\mu, A_\nu\right]$ for the non-Abelian 1-form $A^{(1)} = dx_\mu A^\mu_a\, T^a$ gauge field $A_\mu = A_{a\mu} T^a$ where $d = dx^\mu \partial_\mu$ is the exterior derivative (with $d^2 = 0$). Similarly, another 2-form $G^{(2)} = d\phi^{(1)} - i\left(A^{(1)} \wedge \phi^{(1)}\right) - i\left(\phi^{(1)} \wedge A^{(1)}\right) = \frac{1}{2!}\left(dx^\mu \wedge dx^\nu\right) G_{a\mu\nu}\, T^a$ defines the curvature tensor $G_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu$ corresponding to 1-form $\phi^1 = dx^\mu \phi_{a\mu}\, T^a$ vector field $\phi_\mu = \phi_{a\mu} T^a$. In the above, the vector fields A_μ and ϕ_μ have opposite parity, thus the JP model becomes parity invariant. In the this classical theory in commutative spacetime the fields are Lie algebra-valued $\Psi = \Psi^a T^a$ but in the noncommutative spacetime for an arbitrary gauge group, as it was mentioned before, this property will be lost.

This theory is invariant under the non-Abelian transformation

$$\delta_\theta A_\mu = D_\mu \theta \tag{6.2}$$

$$\delta_\theta \phi_\mu = -i[\phi_\mu, \theta]. \tag{6.3}$$

The Lie algebra of the generators for the symmetry group of A_μ is given by

$$\left[Q^a, Q^b\right] = i f^{abc} Q^c \tag{6.4}$$

and we recall that the vector potential $A_{a\mu}$ is the connection associated with this group. The gauge group of ϕ_μ is Abelian and its generators are symmetric matrices with the same number of generators as A_μ and they obey the following commutation relationship

$$\left[P^a, P^b\right] = 0. \tag{6.5}$$

Also, it is assumed that the generators of these two algebra satisfy the following relation

$$\left[Q^a, P^b\right] = i f^{abc} P^c. \tag{6.6}$$

In the case of $su(n)$, the generators of Lie algebra are traceless and Hermitian matrices, also we will assume that the generators P^a are symmetric matrices.

By turning the coupling off,

$$\mathcal{S}_q \equiv \mathcal{S}(\text{coupling constant=0}) \tag{6.7}$$

the action (6.1) reduces to an action which is invariant under two different Abelian transformations

$$\begin{aligned}\delta_{q1}A_\mu &= \partial_\mu\theta \quad ; \quad \delta_{q1}\phi_\mu = 0\\ \delta_{q2}A_\mu &= 0 \quad ; \quad \delta_{q2}\phi_\mu = \partial_\mu\xi.\end{aligned} \tag{6.8}$$

For Green functions generating functional (or in partition function) we just need the gauge fixing terms for its gauge symmetries (6.2). However, the propagators will be calculated in terms of a quadratic action (6.7) which still possesses the gauge symmetry (6.8). i.e. gauge fixing of the non-Abelian action will not be enough to eliminate the superficial fields in (6.7) which is essential to define finite propagators.

A general quantization procedure of the theories whose gauge symmetries are in the quadratic and the full cases are not consistent is not available yet [113]. Jackiw and Pi proposed to enlarge the configuration space by introducing the new fields ρ and to deal with the action (Extended JP)

$$\mathcal{S}_{ext} = Tr\int d^3x\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(G^{\mu\nu} - i\left[F^{\mu\nu},\rho\right]\right)\left(G_{\mu\nu} - i\left[F_{\mu\nu},\rho\right]\right) - m\epsilon^{\mu\nu\rho}F_{\mu\nu}\phi_\rho\right) \tag{6.9}$$

which is invariant under two different type of non-Abelian transformations

$$\text{Yang-Mills}\begin{cases}\delta_\theta A_\mu = D_\mu\theta\\ \delta_\theta\phi_\mu = -i[\phi_\mu,\theta]\\ \delta_\theta\rho = -i[\rho,\theta]\end{cases} \tag{6.10}$$

and

$$\text{Non-Yang-Mills}\begin{cases}\delta_\chi A_\mu = 0\\ \delta_\chi\phi_\mu = D_\mu\chi\\ \delta_\chi\rho = -\chi\end{cases} \tag{6.11}$$

The additional scalar field ρ transforms under the first gauge transformation as an adjoint vector while the second one applies a shift.

In this thesis we just study the NC version of JP model and postpone the analysis of the JP-extended for future works.

6.2 Noncommutative Jackiw-Pi model

The NC version of original JP model will be written as

$$\hat{\mathcal{S}} = Tr\int d^3x\left\{\frac{1}{2}\hat{F}^{\mu\nu}\star\hat{F}_{\mu\nu} + \frac{1}{2}\hat{G}^{\mu\nu}\star\hat{G}_{\mu\nu} - m\epsilon^{\mu\nu\rho}\hat{F}_{\mu\nu}\star\hat{\phi}_\rho\right\}. \tag{6.12}$$

In similarity with commutative spacetime, the following definitions in the NC space are claimed as

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu,\hat{A}_\nu]_\star \tag{6.13}$$

$$\hat{G}_{\mu\nu} = \hat{D}_{\mu}\hat{\phi}_{\nu} - \hat{D}_{\nu}\hat{\phi}_{\mu} \tag{6.14}$$

$$\hat{D}_{\mu}\hat{\phi}_{\nu} = \partial_{\mu}\hat{\phi}_{\nu} - i[\hat{A}_{\mu}, \hat{\phi}_{\nu}]_{\star} \tag{6.15}$$

where $[A, B]_{\star} = A \star B - B \star A$ as before. By using the definition of Moyal-Weyl star product, up to the first order, we have

$$[A, B]_{\star} = [A, B] + \frac{i}{2}\theta^{ij}\{\partial_i A, \partial_j B\}. \tag{6.16}$$

It is worthy to mention again that in a general noncommutative spacetime the objects inside the above anticommutator take value in the universal enveloping algebra, $\mathcal{U}\left(su(n)\right)$. According to the SW map the gauge transformations are form-invariant, just the fields and operators must be reformulated in NC spacetime. In the other words:

$$\begin{cases} \delta_{\theta}\hat{A}_{\mu} = \hat{D}_{\mu}\hat{\theta} = \partial_{\mu}\hat{\theta} - i[\hat{A}_{\mu}, \hat{\theta}]_{\star} \\ \delta_{\theta}\hat{\phi}_{\mu} = -i[\hat{\phi}_{\mu}, \hat{\theta}]_{\star} \end{cases} \tag{6.17}$$

and

$$\begin{cases} \delta_{\chi}\hat{A}_{\mu} = 0 \\ \delta_{\chi}\hat{\phi}_{\mu} = \hat{D}_{\mu}\hat{\chi} = \partial_{\mu}\hat{\chi} - i[\hat{A}_{\mu}, \hat{\chi}]_{\star} \end{cases}. \tag{6.18}$$

The action has three parts that must be mapped to commutative spacetime. The Yang-Mills term, dynamical/interaction term of ϕ_{μ} and the third one is a Chern-Simons like term. As we saw earlier, the SW map gives us a way to express the variables of NC spacetime in terms of commutative ones up to some freedom. Mapping of the Yang-Mills, term up to the first order, is driven by integration of relation (3.37) and the result is [37]

$$\begin{aligned} \frac{1}{2}Tr\int \hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu} d^3x &= \frac{1}{2}Tr\int \hat{F}^{\mu\nu}\hat{F}_{\mu\nu} d^3x \\ &= \frac{1}{2}Tr\int d^3x \left(F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\theta^{kl}F_{kl}F_{\mu\nu}F^{\mu\nu} + \theta^{kl}F_{\mu k}F_{\nu l}F^{\mu\nu}\right). \end{aligned} \tag{6.19}$$

The vector field ϕ_{μ} transforms in adjoint representation of the gauge group. So the SW map tells us that, up to the first order, this field can be expressed as

$$\begin{aligned} \hat{\phi}_{\mu} &= \phi_{\mu} - \frac{1}{4}\theta^{\rho\sigma}\{A_{\rho}, \partial_{\sigma}\phi_{\mu} + D_{\sigma}\phi_{\mu}\} \\ &\equiv \phi_{\mu} + \theta\phi^{1}_{\mu} \end{aligned} \tag{6.20}$$

where $D_{\mu}\bullet = \partial_{\mu}\bullet - i[A_{\mu}, \bullet]$.

The second term of action (6.12) is more complicated and needs more attention. Using the SW map this term can be written as

$$\begin{aligned}\frac{1}{2}Tr\int d^3x\,\hat{G}^{\mu\nu}\star\hat{G}_{\mu\nu} &= \frac{1}{2}Tr\int d^3x\Big(\hat{D}^{\mu}\hat{\phi}^{\nu}-\hat{D}^{\nu}\hat{\phi}^{\mu}\Big)\star\Big(\hat{D}_{\mu}\hat{\phi}_{\nu}-\hat{D}_{\nu}\hat{\phi}_{\mu}\Big) \\ &= \frac{1}{2}Tr\int d^3x\Big(\hat{D}^{\mu}\phi^{\nu}+\hat{D}^{\mu}\phi^{1\nu}-\hat{D}^{\nu}\phi^{\mu}-\hat{D}^{\nu}\phi^{1\mu}\Big) \\ &\qquad\star\Big(\hat{D}_{\mu}\phi_{\nu}+\hat{D}_{\mu}\phi^{1}_{\nu}-\hat{D}_{\nu}\phi_{\mu}-\hat{D}_{\nu}\phi^{1}_{\mu}\Big).\end{aligned} \tag{6.21}$$

The covariant derivative in the above expression is given by

$$\hat{D}_{\mu}\phi_{\nu}=D_{\mu}\phi_{\nu}-i\Big[A^{1}_{\mu},\phi_{\nu}\Big]+\frac{\theta^{\alpha\beta}}{2}\Big\{\partial_{\alpha}A_{\mu},\partial_{\beta}\phi_{\nu}\Big\} \tag{6.22}$$

where A^{1}_{μ} is the first term of the expansion of NC field $\hat{A}_{\mu}$ in terms of commutative fields, as we saw in (3.35). By plugging in the expanded covariant derivative in (6.21) we obtain

$$\begin{aligned}&\frac{1}{2}Tr\int d^3x(\hat{G}^{\mu\nu})\star(\hat{G}_{\mu\nu}) \\ =\;& \frac{1}{2}Tr\int d^3x\Big(D^{\mu}\phi^{\nu}D_{\mu}\phi_{\nu}-iD^{\mu}\phi^{\nu}\Big[A^{1}_{\mu},\phi_{\nu}\Big]+\frac{\theta^{\alpha\beta}}{2}D^{\mu}\phi^{\nu}\Big\{\partial_{\alpha}A_{\mu},\partial_{\beta}\phi_{\nu}\Big\}+D^{\mu}\phi^{\nu}D_{\mu}\phi^{1}_{\nu} \\ -\;& D^{\mu}\phi^{\nu}D_{\nu}\phi_{\mu}-iD^{\mu}\phi^{\nu}\Big[A^{1}_{\nu},\phi_{\mu}\Big]+\frac{\theta^{\alpha\beta}}{2}D^{\mu}\phi^{\nu}\Big\{\partial_{\alpha}A_{\nu},\partial_{\beta}\phi_{\mu}\Big\}-D^{\mu}\phi^{\nu}D_{\nu}\phi^{1}_{\mu} \\ +\;& D^{\mu}\phi^{1\nu}D_{\mu}\phi_{\nu}-D^{\mu}\phi^{1\nu}D_{\nu}\phi_{\mu}-i\Big[A^{1\mu},\phi^{\nu}\Big]D_{\mu}\phi_{\nu}+i\Big[A^{1\mu},\phi^{\nu}\Big]D_{\nu}\phi_{\mu} \\ +\;& \frac{\theta^{\alpha\beta}}{2}\Big\{\partial_{\alpha}A^{\mu},\partial_{\beta}\phi^{\nu}\Big\}D_{\mu}\phi_{\nu}-\frac{\theta^{\alpha\beta}}{2}\Big\{\partial_{\alpha}A^{\mu},\partial_{\beta}\phi^{\nu}\Big\}D_{\nu}\phi_{\mu} \\ -\;& D^{\nu}\phi^{\mu}D_{\mu}\phi_{\nu}+iD^{\nu}\phi^{\mu}\Big[A^{1}_{\mu},\phi_{\nu}\Big]-\frac{\theta^{\alpha\beta}}{2}D^{\nu}\phi^{\mu}\Big\{\partial_{\alpha}A_{\mu},\partial_{\beta}\phi_{\nu}\Big\}-D^{\nu}\phi^{\mu}D_{\mu}\phi^{1}_{\nu} \\ +\;& D^{\nu}\phi^{\mu}D_{\nu}\phi_{\mu}+iD^{\nu}\phi^{\nu}\Big[A^{1}_{\nu},\phi_{\mu}\Big]-\frac{\theta^{\alpha\beta}}{2}D^{\nu}\phi^{\mu}\Big\{\partial_{\alpha}A_{\nu},\partial_{\beta}\phi_{\mu}\Big\}+D^{\nu}\phi^{\mu}D_{\nu}\phi^{1}_{\mu}.\end{aligned} \tag{6.23}$$

After doing some algebra the above expression can be simplified as

$$\begin{aligned}&\frac{1}{2}Tr\int d^3x(\hat{G}^{\mu\nu})\star(\hat{G}_{\mu\nu})= \\ &\frac{1}{2}Tr\int d^3x\,\left(G^{\mu\nu}G_{\mu\nu}+3G^{1}_{\mu\nu}D^{\mu}\phi^{\nu}-iG_{\mu\nu}\Big[A^{1\mu},\phi^{\nu}\Big]+\frac{\theta^{\alpha\beta}}{2}G^{\mu\nu}\Big\{\partial_{\alpha}A_{\mu},\partial_{\beta}\phi_{\nu}\Big\}\right)\end{aligned} \tag{6.24}$$

where $G^{1}_{\mu\nu}=D_{\mu}\phi^{1}_{\nu}-D_{\nu}\phi^{1}_{\mu}$. The above expression can be rewritten solely in terms of ordinary fields of commutative theory,

$$\begin{aligned}\frac{1}{2}Tr\int d^3x(\hat{G}^{\mu\nu})\star(\hat{G}_{\mu\nu}) &= \frac{1}{2}Tr\int d^3x\,\Big(G^{\mu\nu}G_{\mu\nu} \\ &- 3\theta^{\rho\sigma}G^{\mu\nu}\Big(D_{\mu}A_{\rho}\left(\partial_{\sigma}+D_{\sigma}\right)\phi_{\nu}-\frac{1}{3}\partial_{\alpha}A_{\mu}\partial_{\beta}\phi_{\nu}\Big).\end{aligned} \tag{6.25}$$

According to the SW map, the Chern-Simons like term can be transformed as follows

$$\begin{aligned}mTr\int d^3x\,\epsilon^{\mu\nu\rho}\hat{F}_{\mu\nu}\star\hat{\phi}_{\rho} &= mTr\int d^3x\,\epsilon^{\mu\nu\rho}\hat{F}_{\mu\nu}\hat{\phi}_{\rho} \\ &= mTr\epsilon^{\mu\nu\rho}\int d^3x\,\Big(F_{\mu\nu}+F^{1}_{\mu\nu}\Big)\Big(\phi_{\rho}+\phi^{1}_{\rho}\Big) \\ &= mTr\epsilon^{\mu\nu\rho}\int d^3x\,\Big(F_{\mu\nu}\phi_{\rho}+F^{1}_{\mu\nu}\phi_{\rho}+F_{\mu\nu}\phi^{1}_{\rho}\Big).\end{aligned} \tag{6.26}$$

This expression also can be rewritten just in terms of variables of the original theory

$$
\begin{aligned}
mTr\int d^3x\,\epsilon^{\mu\nu\rho}\hat{F}_{\mu\nu}\star\hat{\phi}_\rho &= mTr\epsilon^{\mu\nu\rho}\int d^3x\,\Big(F_{\mu\nu}\phi_\rho \\
&+ \theta^{\alpha\beta}\Big(F_{\mu\alpha}F_{\nu\beta}\phi_\rho+\frac{1}{4}F_{\mu\nu}\Big\{\phi_\rho,(\partial_\beta+D_\beta)A_\alpha\Big\}\Big)\Big).
\end{aligned} \tag{6.27}
$$

The noncommutative JP theory is given by adding up Eqs.(6.19), (6.25) and (6.27)

$$
\begin{aligned}
\hat{\mathcal{S}} &= Tr\int d^3x\left\{\frac{1}{2}\hat{F}^{\mu\nu}\star\hat{F}_{\mu\nu}+\frac{1}{2}\hat{G}^{\mu\nu}\star\hat{G}_{\mu\nu}-m\epsilon^{\mu\nu\rho}\hat{F}_{\mu\nu}\star\hat{\phi}_\rho\right\} \\
&= \mathcal{S}+\frac{1}{2}Tr\int d^3x\Big(-\frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}+\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} \\
&- 3\theta^{\rho\sigma}G^{\mu\nu}\Big(D_\mu A_\rho\left(\partial_\sigma+D_\sigma\right)\phi_\nu-\frac{1}{3}\partial_\alpha A_\mu\partial_\beta\phi_\nu\Big) \\
&+ \theta^{\alpha\beta}\Big(F_{\mu\alpha}F_{\nu\beta}\phi_\rho+\frac{1}{4}F_{\mu\nu}\Big\{\phi_\rho,(\partial_\beta+D_\beta)A_\alpha\Big\}\Big)\Big)
\end{aligned} \tag{6.28}
$$

This complete $\mathcal{O}\left(\theta^1\right)$ noncommutative JP theory contains vertices, with a higher number of gauge bosons, that are absent in the original theory and from the phenomenological point of view these two Lagrangian produce different interactions. We have not included structure constants explicitly in our analysis so one can not discuss the pertubation expansion of the NC theory. For the future work, we are going to add fermionic matter field in the theory with explicit structure constants and we will analyze its perturbative expansion and the phenomenological aspects of both theories.

7 The field-antifield treatment of extended Jackiw-Pi model

7.1 A Fast Review of Field-Antifield (or Batalin-Vilkovisky) Formalism

The basic idea of the so-called Field-Antifield formalism is to generalize the BRST invariance to the theories with arbitrary gauge structure. The ingredients are the ordinary fields Φ^A, the ghosts, the auxiliary fields and their canonically conjugated antifields $\Phi^\star_A$. With all these elements we can construct the well-known Field-Antifield or Batalin-Vilkovisky (BV) action. At the classical level, the BV action becomes the ordinary classical action when all the antifields are zeroed. A gauge-fixed action can be obtained by a canonical transformation. At this time we can say that the action is in a gauge-fixed basis. The other way to fix the gauge is through the choice of a gauge fermion and to make the antifields to be equal to the functional derivative of this fermionic function.
This method can be applied to gauge theories which have an open algebra (the algebra of gauge transformations closes only on shell), to closed algebras, to gauge theories that have structure functions rather than constants (soft algebras), and to the case where the gauge transformations may or may not be independent, reducible or irreducible algebras respectively. Zinn-Justin introduced the concept of sources of BRST-transformations [114]. These sources are the antifields in the BV formalism. It was shown also that the geometry of the antifields have a natural origin [115].

At the quantum level, the field-antifield formalism also works at one-loop anomalies [116, 117]. Here, with the addition of extra degrees of freedom, which leads to an extension of the original configuration space, we have a solution for the regularized quantum master equation (QME) at one-loop that has been obtained as an independent part of the antifields inside the anomaly.

7.1.1 Gauge structure

In a gauge theory the action is invariant under a set of gauge transformations with infinitesimal form

$$\delta\Psi^i(x) = (R^i_\alpha \varepsilon^\alpha)(x) \tag{7.1}$$

where $i = 1, 2, \cdots n$ is the number of fields, $\alpha = 1, 2, \cdots m < n$ is the number of sets of gauge transformations and and R^i_α are the generators of gauge transformations. The ε^α are the infinitesimal gauge parameters and R^i_α the generators of the gauge transformations. When $\epsilon_\alpha = \epsilon(\varepsilon^\alpha) = 0$ we have an ordinary symmetry, when $\epsilon_\alpha = 1$ the equation is characteristic of a supersymmetry. The Grassmann parity of generators of the gauge transformations is defined as $\epsilon(R^i_\alpha) = \epsilon_\alpha + \epsilon_i$. Also we have $\epsilon_i = \epsilon(\phi^i)$ that defines Grassmann parity of the fields. Fields with $\epsilon_i = 0$ are called bosonic and with $\epsilon_i = 1$ are fermionic. The relation (7.1) is written in the DeWitt compact notation and its original form is

$$\delta\Psi^i(x) = \sum_\alpha \int dy\, R^i_\alpha(x,y)\epsilon^\alpha(y) \tag{7.2}$$

The graded commutation rule is defined as

$$\phi^i\,(x)\,\phi^j\,(y) = (-1)^{\epsilon_i\epsilon_j}\,\phi^j\,(y)\,\phi^i\,(x)\,. \tag{7.3}$$

Let $S_{0,i}(\phi,x)$ denote the variation of the action with respect to $\phi^i(x)$:

$$S_{0,i} \equiv \frac{\partial_r S_0[\phi]}{\partial\phi^i(x)} \tag{7.4}$$

where the subscript i after the comma denotes the right derivative with respect to the corresponding field, that is, the field is to be commutated to the far right and then dropped. When using right derivatives, the variation δS_0 of S_0, or of any other object, is given by $\delta S_0 = S_{0,i}\delta\phi^i$. If one were to use left derivatives, the variation of S_0 would be read $\delta S_0 = \delta\phi^i\frac{\partial_l S_0}{\partial\phi^i}$. The commutation rule for the gauge transformations in the most general form obeys the following relationship

$$[\delta_1,\delta_2]\,\phi^i = \left(R^i_\gamma T^\gamma_{\alpha\beta} - S_{0,j}E^{ij}_{\alpha\beta}\right)\varepsilon^\beta_1\varepsilon^\alpha_2 \tag{7.5}$$

where the tensors $T^\gamma_{\alpha\beta}$ are called the structure constants of the gauge algebra, although they depend, in general, on the fields of the theory. When $E^{ij}_{\alpha\beta} = 0$, the gauge algebra is said to be closed, otherwise it is open. Equation (7.5) defines a Lie algebra if the algebra is closed and the $T^\gamma_{\alpha\beta}$ are independent of the fields. We will see that the Jackiw-Pi model has a closed and Lie algebraic gauge structure.

When we say that the action is invariant under the gauge transformation in Eq.(7.1) means that the Noether identities

$$\int dx\,\sum_{i=1}^n S_{0,i}(x)R^i_\alpha(x,y) = 0 \tag{7.6}$$

hold, or equivalently, in compact notation

$$S_{0,i}R^i_\alpha = 0. \tag{7.7}$$

Hence the field equations may be written as

$$S_{0,i} = 0. \tag{7.8}$$

As in the familiar Faddeev-Popov procedure, it is useful to introduce ghost fields C with opposite Grassmann parities to the gauge parameters ε^α

$$\epsilon\,(C^\alpha) = \epsilon_\alpha + 1\,(mod \quad 2) \tag{7.9}$$

and to replace the gauge parameters by ghost fields.

7.1.2 Irreducible and reducible gauge theories

It is important to know any dependences among the gauge generators. After analyzing these relations it is possible to determine the independent degrees of freedom. The simplest gauge theories, for which all gauge transformations are independent, are called irreducible. When dependences exist, the theory is reducible. In reducible gauge theories, there is a “kind of gauge invariance for gauge transformations” or what one might call “level-one”gauge invariances. If the level-one gauge transformations are independent, then the theory is called first-stage reducible. This may not happen. Then, there are “level-two” gauge invariances, i.e., gauge invariances for the level-one gauge invariances and so on. This leads to the concept of an L-th stage reducible theory. In what follows we let m s denote the number of gauge generators at the s-th stage regardless of whether they are independent.

In this brief review we will consider only theories with irreducible gauge structure. For more detailed discussion of the full formalism the interested reader is encouraged to see [118, 50].

7.1.3 Introducing the antifields

We incorporate the ghost fields into the field set $\Psi^A = \{\phi^i, C^\alpha\}$, where $i = 1, ..., n$ and $\alpha = 1, ..., m$. We call it a *minimal* set. Clearly $A = 1, ..., N$, where $N = n + m$. One then further increases the set by introducing an antifield $\Psi^\star_A$ for each field Ψ^A. The Grassmann parity of the antifields is $\epsilon\left(\Psi^\star_A\right) = \epsilon\left(\Psi^a\right) + 1 \, (mod \quad 2)$.

We assign a new number to each field, the ghost number **gh**, which is defined as follow

$$\begin{aligned}
\mathbf{gh}\left[\phi^i\right] &= 0 \\
\mathbf{gh}\left[C^\alpha\right] &= 1 \\
\mathbf{gh}\left[\Psi^\star_A\right] &= -gh\left[\Psi_A\right] - 1.
\end{aligned}$$

In this generalized space, the antibracket is defined by

$$(X, Y) = \frac{\partial_r X}{\partial \Psi^A} \frac{\partial_l Y}{\partial \Psi^\star_A} - \frac{\partial_r X}{\partial \Psi^\star_A} \frac{\partial_l Y}{\partial \Psi^A} \tag{7.10}$$

where ∂_r denotes the right derivative and ∂_l the left derivative. The antibracket is graded antisymmetric

$$(X, Y) = -\left(-1\right)^{(\epsilon_X+1)(\epsilon_Y+1)} (Y, X)\,. \tag{7.11}$$

If one groups the fields and the antifields together into the set

$$z^a = \left\{\Psi^\star_A, \Psi^A\right\} \qquad a = 1, 2, .., 2N \tag{7.12}$$

then the antibracket is seen to define a symplectic structure on the space of fields and antifields

$$(X, Y) = \frac{\partial_r X}{\partial z^a} \omega^{ab} \frac{\partial_l Y}{\partial z^b} \tag{7.13}$$

with

$$\omega^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}. \tag{7.14}$$

The antifield can be thought of as a kind of conjugate variable to the field, since

$$\left(\Psi^A, \Psi^\star_B\right) = \delta^A_B. \tag{7.15}$$

As it can be seen the antibracket is, in some sense, very similar to the Poisson bracket in the phase-space. In fact, by introducing the antifields and defining the antibracket we have an odd(even) symplectic structure inside the Lagrangian formalism. In this way, we can enjoy the clarity and power of Hamiltonian formalism right inside the extended configuration space.

The antibracket of two fermionic fields is

$$(F, F) = 0, \tag{7.16}$$

for two bosonic fields is

$$(B, B) = 2\frac{\partial B}{\partial \Psi^A}\frac{\partial B}{\partial \Psi^\star_B} \tag{7.17}$$

and for any field X, the triple commutation gives

$$(X\,(X, X)) = 0. \tag{7.18}$$

7.1.4 The classical master equation

Let $S\left[\Psi^A, \Psi^\star_B\right]$ be a functional of the fields and antifields with the dimension of an action, vanishing ghost number and even Grassmann parity. The equation

$$(S, S) = 2\frac{\partial S}{\partial \Psi^A}\frac{\partial S}{\partial \Psi^\star_A} = 0 \tag{7.19}$$

is the classical master equation. The solutions of the classical master equation with suitable boundary conditions turn out to be generating functionals for the gauge structure of the theory. S is also the starting point for the quantization.

Finally, the action $S\left[\Psi^A, \Psi^\star_B\right]$ can be expanded in a series in the antifields, while maintaining vanishing ghost number and even Grassmann parity

$$\begin{aligned} S_{BV} = S\left[\Psi^A, \Psi^\star_B\right] &= S_0 + \phi^\star_i R^i_\alpha C^\alpha + C^\star_\alpha \frac{1}{2} T^\alpha_{\beta\gamma} \left(-1\right)^{\epsilon_\beta} C^\gamma C^\beta \\ &+ \phi^\star_i \phi^\star_j \left(-1\right)^{\epsilon_i} \frac{1}{4} E^{ji}_{\alpha\beta} \left(-1\right)^{\epsilon_\alpha} C^\beta C^\alpha. \end{aligned}$$

When this is inserted into the classical master equation, one finds that this equation implies the gauge structure of the classical theory. In fact, this form is not unique but is the brief one for S_{BV}. One can turn back to the classical action S_0 when the antifields go to zero

$$S_{BV}[\Psi, \Psi^\star]\big|_{\Psi^\star=0} = S_0[\phi]. \tag{7.20}$$

7.1.5 Gauge Fixing and Quantization

Although ghost fields have been incorporated into the theory, the solutions of classical master equation (7.19) have a set of invariances

$$\frac{\partial S}{\partial z^a} R^a_b = 0, \tag{7.21}$$

with

$$R^a_b = \omega^{ac} \frac{\partial_l \partial_r S}{\partial z^c \partial z^b}. \tag{7.22}$$

Due to these gauge freedoms the action (7.20), as a solution of classical master equation is not suitable for quantization via path integral and a gauge-fixing procedure is needed. The theory also contains many antifields that usually one wants to eliminate before computing amplitudes and S-matrix elements. One cannot simply set the antifields to zero because the action would reduce to the original classical action S_0 , which is not appropriate for starting perturbation theory due to gauge invariances. In the Batalin– Vilkovisky approach the gauge is fixed using a fermionic function which has Grassmann parity $\epsilon(\Theta)$=1, $\mathbf{gh}[\Theta] = -1$ and is functional of fields Ψ^A only. The antifields are eliminated through relation

$$\Psi^\star_A = \frac{\partial \Theta}{\partial \Psi^A} \tag{7.23}$$

After implementing this gauge-fixing procedure we can define a surface in the functional space

$$\Sigma_\Theta = \left\{ \left(\Psi^A, \Psi^\star_A \right) | \Psi^\star_A = \frac{\partial \Theta}{\partial \Psi^A} \right\}. \tag{7.24}$$

Hence for any functional $X\left[\Phi, \Phi^\star\right]$ we have

$$X|_{\Sigma_\Theta} = X \left[\Psi, \frac{\partial \Psi}{\partial \Phi} \right] \tag{7.25}$$

To construct a gauge-fixing fermion Θ of ghost number -1, one must again introduce additional auxiliary fields. The simplest choice utilizes a trivial pair $\bar{C}^\alpha$ and $\bar{\pi}^\alpha$ with the following properties

$$\begin{aligned} \epsilon\left(\bar{C}^\alpha\right) &= \epsilon_\alpha + 1, \quad \epsilon\left(\bar{\pi}^\alpha\right) = \epsilon_\alpha \\ \mathbf{gh}\left[\bar{C}^\alpha\right] &= -1, \qquad \mathbf{gh}\left[\bar{\pi}^\alpha\right] = 0. \end{aligned} \tag{7.26}$$

The auxiliary fields $\bar{C}_\alpha$ are the Faddeev-Popov antighosts ($\bar{\pi}^\alpha$ are called Nakanishi-Lautrup fields)[6]. Along with these fields we include the corresponding antifields $\bar{C}^\star_\alpha$ and $\bar{\pi}^\star_\alpha$. Adding the term $\bar{\pi}^\alpha \bar{C}^\star_\alpha$ to the action S does not spoil its properties as a proper solution to the classical master equation, and one obtains the *non-minimal* action

$$S_{nm} = S + \bar{\pi}^\alpha \bar{C}^\star_\alpha. \tag{7.27}$$

[6] Do not confuse antighost with anti-ghost.

We can think of these new auxiliary fields as a kind of Lagrange multipliers for the gauge-fixing terms. The simplest possibility for fermionic function Θ is

$$\Theta = \bar{C}^{\alpha} \chi_{\alpha} (\phi) \tag{7.28}$$

where χ_{α} are the gauge-fixing conditions for the fields ϕ. The gauge-fixed action is denoted by

$$S_{\Theta} = S_{BV-nm}|_{\Sigma_{\Theta}}. \tag{7.29}$$

The quantum generating functional is defined by using the constraint (7.23) to calculate the correlation function X as

$$I|_{\Theta} (X) = \int \mathcal{D}\Psi \mathcal{D}\Psi^{\star} \delta \left(\Psi^{\star}_{A} - \frac{\partial \Theta}{\partial \Psi^{A}} \right) e^{\frac{i}{\hbar} W[\Psi, \Psi^{\star}]} X [\Psi, \Psi^{\star}] . \tag{7.30}$$

Here W is the quantum action, which reduces to S in the limit $\hbar \to 0$. An admissible Θ leads to well- defined propagators when the path integral is expressed as a perturbation series expansion. For a detailed discussion of the W we refer the interested reader to the references [50, 118].

7.2 Field-antifield treatment of extended Jackiw-Pi model

According to gauge transformations, Eqs.(6.10) and (6.11) the gauge structure of extended JP model can be expressed in a compact form $\delta \Psi^{i} = R^{i}_{\alpha} \varepsilon^{\alpha}$ or

$$\begin{pmatrix} \delta A_{\mu} \\ \delta \phi_{\mu} \\ \delta \rho \end{pmatrix} = \begin{pmatrix} D_{\mu} & 0 \\ [\phi_{\mu}, \circ] & D_{\mu} \\ [\rho, \circ] & -1 \end{pmatrix} \begin{pmatrix} \theta \\ \chi \end{pmatrix} . \tag{7.31}$$

The dynamical variables of the model, i.e. A_{μ}, ϕ_{μ} and ρ are bosonic fields so their Grassmann parity is $\epsilon_{i} = 0$. The gauge parameters θ and χ also are bosonic variables hence their Grassmann parity is $\epsilon_{\alpha} = 0$.

For the first step we have to calculate the commutation of two gauge transformations. For the gauge field A_{μ} we have

$$[\delta_1, \delta_2] A^{a}_{\mu} = \partial_{\mu} \theta^{a}_{12} + f^{ade} A^{d}_{\mu} \theta^{e}_{12} = D^{ae}_{\mu} \theta^{e}_{12} \tag{7.32}$$

where $\theta^{e}_{12} = f^{ecb} \theta^{c}_{1} \theta^{b}_{2}$. For the vector field ϕ_{μ} one finds

$$[\delta_1, \delta_2] \phi^{a}_{\mu} = f^{adb} \phi^{d}_{\mu} \left(f^{bec} \theta^{e}_{1} \theta^{c}_{2} \right) + D^{ad}_{\mu} \left(f^{dbc} \chi^{b}_{1} \theta^{c}_{2} + f^{dcb} \theta^{c}_{1} \chi^{b}_{2} \right) \tag{7.33}$$

Additionally for the scalar field ρ, we yield

$$[\delta_1, \delta_2] \rho^{a} = f^{adb} \rho^{d} \left(f^{bec} \theta^{e}_{1} \theta^{c}_{2} \right) - \left(f^{abc} \chi^{b}_{1} \theta^{c}_{2} + f^{acb} \theta^{c}_{1} \chi^{b}_{2} \right) . \tag{7.34}$$

As we can see from the commutations of fields, the gauge algebra of the extended JP model is closed and all of $E^{ij}_{\alpha\beta}$ are equal to zero. In other words, there is not any term dependent on the equation of motion. The next step would be determine the structure constants of the gauge algebra according to Eq.(7.5). As an interesting result we find that the non-zero structure constant of all the above commutations are the same and are equal to $T^{\alpha}_{\beta\gamma} = f^{abc}$.

Now we have the enough ingredients to construct the field-antifield action for the theory at hand as

$$\begin{aligned} S_{BV} &= S_0 + A^{\star a}_{\mu} D^{\mu ab}\xi^b + \phi^{\star a}_{\mu}\left(f^{abc}\phi^{\mu b}\xi^c + D^{\mu ab}\eta^b\right) + \rho^{\star a}\left(f^{abc}\rho^b\xi^c - \eta^a\right) &(7.35)\\ &+ \eta^{\star a} f^{abc}\xi^c\xi^b + \xi^{\star a} f^{abc}\eta^c\xi^b &(7.36)\end{aligned}$$

where ξ and η are ghost fields related to the gauge parameters θ and χ, respectively. The Grassmann parity of these ghosts is $\epsilon(\xi) = \epsilon(\eta) = 1$. The ghost numbers of the variables of action S_{BV} are

$$\begin{aligned} &\mathbf{gh}[A_\mu] = \mathbf{gh}[\phi_\mu] = \mathbf{gh}[\rho] = 0, && \mathbf{gh}[\xi] = \mathbf{gh}[\eta] = 1,\\ &\mathbf{gh}\left[A^\star_\mu\right] = \mathbf{gh}\left[\phi^\star_\mu\right] = \mathbf{gh}[\rho^\star] = -1, && \mathbf{gh}[\xi^\star] = \mathbf{gh}[\eta^\star] = -2. \end{aligned} \qquad (7.37)$$

Before moving forward to the quantum world we have to fix the gauge degrees of freedom. To realize this we go to a gauge-fixed basis by introducing a fermionic function with the ghost number equal to $\mathbf{gh}[\Theta] = -1$ and Grassmann parity $\epsilon(\Theta) = -1$, as mentioned before. Without lost of generality we suggest the fermionic function

$$\Theta = \int d^3x\, \bar{\xi}^a\left(-\frac{\bar{\pi}^a}{2\gamma} + \partial^\mu A^a_\mu\right) + \bar{\eta}^a\left(-\frac{\bar{\omega}^a}{2\gamma'} + \partial^\mu\phi^a_\mu\right) \qquad (7.38)$$

where $\bar{\xi}^a$ and $\bar{\eta}^a$ are Faddeev-Popov antighost fields related to the ghosts ξ^a and η^a with statistics and ghost number equal to

$$\epsilon\left(\bar{\xi}^a\right) = \epsilon(\bar{\eta}^a) = 1, \qquad \mathbf{gh}\left[\bar{\xi}^a\right] = \mathbf{gh}[\bar{\eta}^a] = -1. \qquad (7.39)$$

It should be mentioned that the final result of a quantization is independent of gauge fixing. Together with the Faddeev-Popov antighost, we have introduced the Nakanishi-Lautrup fields $(\bar{\pi}^a, \bar{\omega}^a)$ to our minimal set to eliminate antighost fields with the following properties

$$\epsilon(\bar{\omega}^a) = \epsilon(\bar{\pi}^a) = 0, \qquad \mathbf{gh}[\bar{\omega}^a] = \mathbf{gh}[\bar{\pi}^a] = 0. \qquad (7.40)$$

It is necessary to include the antifields associated to these new auxiliary fields with the following properties

$$\begin{aligned} &\epsilon\left(\bar{\xi}^{\star a}\right) = \epsilon(\bar{\eta}^{\star a}) = 0, && \mathbf{gh}\left[\bar{\xi}^{\star a}\right] = \mathbf{gh}[\bar{\eta}^{\star a}] = 0,\\ &\epsilon(\bar{\omega}^{\star a}) = \epsilon(\bar{\pi}^{\star a}) = 1, && \mathbf{gh}[\bar{\omega}^{\star a}] = \mathbf{gh}[\bar{\pi}^{\star a}] = -1. \end{aligned} \qquad (7.41)$$

The minimal set together with these new auxiliary fields constitute the so-called *non-minimal* set. The non-minimal extension of Batalin-Vilkovisky action reads

$$S_{BV-nm} = S_{BV} + \bar{\xi}^{\star a}\bar{\pi}^a + \bar{\eta}^{\star a}\bar{\omega}^a \qquad (7.42)$$

By employing the Gaussian-averaging gauge-fixing procedure we have

$$\Psi^{\star}_{A} = \frac{\partial \Theta}{\partial \Psi^{A}}. \tag{7.43}$$

With this choice we can eliminate the antifields via Eqs.(7.38) and (7.43)

$$\begin{aligned}
&A^{\star a}_{\mu} = -\partial^{\mu}\bar{\xi}^{a}, && \bar{\eta}^{\star a} = -\frac{\bar{\omega}^{a}}{2\gamma'} + \partial^{\mu}\phi^{a}_{\mu},\\
&\phi^{\star a}_{\mu} = -\partial^{\mu}\bar{\eta}^{a}, && \xi^{\star a} = 0,\\
&\bar{\xi}^{\star a} = -\frac{\bar{\pi}^{a}}{2\gamma} + \partial^{\mu}A^{a}_{\mu}, && \eta^{\star a} = 0,\\
&\rho^{\star a} = 0.
\end{aligned} \tag{7.44}$$

Finally we obtain the gauge-fixed quantized-ready action for extended JP model

$$\begin{aligned}
S_{\Theta} =& S_{0} - \int d^{3}x \Bigg(\partial_{\mu}\bar{\xi}^{a} D^{\mu ab}\xi^{b} - \partial_{\mu}\bar{\eta}^{a}\left(f^{abc}\phi^{\mu b}\xi^{c} + D^{\mu ab}\eta^{b}\right)\\
&+ \bar{\pi}^{a}\left(-\frac{\bar{\pi}^{a}}{2\gamma} + \partial^{\mu}A^{a}_{\mu}\right) + \bar{\omega}^{a}\left(-\frac{\bar{\omega}^{a}}{2\gamma'} + \partial^{\mu}\phi^{a}_{\mu}\right)\Bigg)
\end{aligned} \tag{7.45}$$

The Gaussian integration over auxiliary fields $\bar{\pi}$ and $\bar{\omega}$ can be performed for Eq.(7.45) to give

$$\begin{aligned}
S_{\Theta} \longrightarrow -\frac{1}{4}\int d^{3}x \Big(& \frac{1}{2}F^{a\mu\nu}F^{a}_{\mu\nu} + \frac{1}{2}\left(G^{a\mu\nu} - i\left[F^{\mu\nu},\rho\right]^{a}\right)\left(G^{a}_{\mu\nu} - i\left[F_{\mu\nu},\rho\right]^{a}\right) - m\epsilon^{\mu\nu\rho}F^{a}_{\mu\nu}\phi^{a}_{\rho}\\
&+ \partial_{\mu}\bar{\xi}^{a}D^{\mu ab}\xi^{b} - \partial_{\mu}\bar{\eta}^{a}\left(f^{abc}\phi^{\mu b}\xi^{c} + D^{\mu ab}\eta^{b}\right) + \frac{\gamma}{2}\partial^{\mu}A^{a}_{\mu}\partial^{\nu}A^{a}_{\nu} + \frac{\gamma'}{2}\partial^{\mu}\phi^{a}_{\mu}\partial^{\nu}\phi^{a}_{\nu}\Big)
\end{aligned} \tag{7.46}$$

which is very similar to the Yang-Mills action fixed in the R_{γ} gauge. The case $\gamma = \gamma' = 1$ is the Feynman gauge. When $\gamma, \gamma' \to \infty$, the $\bar{\pi}$ and $\bar{\omega}$ dependence in Θ of Eq.(7.38) disappears and the Landau gauge $\partial^{\mu}\phi^{a}_{\mu} = \partial^{\mu}A^{a}_{\mu} = 0$ is imposed as a delta-function condition.

The gauge-fixed BRST transformations are

$$\begin{aligned}
&\delta_{B_{\Theta}}A^{a}_{\mu} = D^{\mu ab}\xi^{b}, && \delta_{B_{\Theta}}\phi^{a}_{\mu} = f^{abc}\phi^{b}_{\mu}\xi^{b} + D^{ab}_{\mu}\eta^{b},\\
&\delta_{B_{\Theta}}\rho^{a} = f^{abc}\rho^{b}\xi^{c} - \eta^{a}, && \delta_{B_{\Theta}}\xi^{a} = f^{abc}\xi^{b}\eta^{c},\\
&\delta_{B_{\Theta}}\eta^{a} = f^{abc}\xi^{b}\eta^{c}, && \delta_{B_{\Theta}}\bar{\xi}^{a} = \bar{\pi}^{a},\\
&\delta_{B_{\Theta}}\bar{\eta}^{a} = \bar{\omega}^{a}, && \delta_{B_{\Theta}}\bar{\pi}^{a} = 0,\\
&\delta_{B_{\Theta}}\bar{\omega}^{a} = 0.
\end{aligned} \tag{7.47}$$

The nilpotency of $\delta_{B_{\Theta}}$ holds off-shell because the original gauge algebra is closed.

The next step would be to discuss the anomalies of this theory and also to calculate its perturbative expansion and anomalies using the above action.

8 Conclusions and perspectives

To investigate some ingredients of the formalism that can lead us to work in the Planck energy scale requires us to discuss the physics of the early Universe, for instance, where quantum mechanics and general relativity were combined and quantum gravity is formed. This is one of the main motivations to study mechanisms that introduces Planck scale parameters in classical systems. As such this is one of the main motivations to use NCy in order to introduce this so-called Planck scale parameter. In this work we have analyzed the free movement of a particle upon a 2-sphere considering a NC classical mechanics approach. In this scenario, we can consider a semi-classical approach where the Planck constant was substituted by the NC parameter.

The NC Newton's second law has shown us that the curvature of the space acted the same way as if there was a potential since the particle flat space acceleration has the NC contribution given by the potential, namely, the NC contribution would be zero but it is not. In the 2-sphere free particle dynamics, the NC additional term is different from zero, which means that its origin is the curvature of the system.

The introduction of NC contribution makes us also ask what would be the nature of the potential effect caused by the curvature. In other words, since in a flat space free particle system the NC contribution is connected with a potential such that if $V = 0$ we have no contribution, and in the curved space this effect does not happen, what is the physical meaning of this potential-type effect brought by the curvature? And in the case of curved space and $V \neq 0$? Where would the NC contribution appear?

Furthermore, we have also introduced some basic ideas of classical mechanics and differential geometry. We started by formulating the procedure of introducing constraints into the Lagrangian formalism: they were inserted via Lagrange multipliers and we have demonstrated that this procedure leads to the same number of degrees of freedom and equations of motion if we had obtained one of the variables of the known constraint and substitute it in the free Lagrangian. After that, we have given a detailed analysis of a particle constrained over a 2-sphere.

Basic notions of differential geometry, such as the metric and Christoffel symbols, appear as a consequence of the description of a constrained Lagrangian system and its corresponding principle of least action. A solution of the equations of motion was given based on geometric grounds and with the help of Noether theorem. It was also shown that physical position variables of the model evolve over an ellipse. We have proposed a central force problem whose solution for position variables are the same as those of the particle over a 2-sphere. One can be led to interpret the curvature of the space where the particle lives as a consequence of an effective potential. This example may be a starting point for studying general relativity. We have also naively discussed the relation between both the

Dirac brackets and Christoffel symbols, since both of them are supposed to describe the correct evolution of a particle constrained to a surface.

Finally, as an example, we treated the so-called *Zitterbewegung* of the Dirac electron. It may be seen as the effective motion of a particle over a 2-sphere, assuming that the electron bears an internal structure.

The extended DFR formulation of NC theories was developed recently and its main mathematical characteristic is to promote the NC parameter, $\theta^{\mu\nu}$, to the status of spacetime coordinates. This procedure recovers the Lorentz invariance of the theory and at the same time it requires the construction of a conjugated momentum associated with $\theta^{\mu\nu}$, together with its respective algebra. In this work, based on the results obtained in two different θ-variable phase-spaces, we have shown, in the DFR spacetime, that this momentum $\pi_{\mu\nu}$ (which completes the set of phase-space symplectic variables as being $(x^\mu, p_\mu, \theta^{\mu\nu}, \pi_{\mu\nu})$) is directly connected to Lorentz invariance and cannot be considered irrelevant in any ordinary DFR analysis since it is essential to calculate the QFT commutation relations for DFR formalism.

Through two examples, the DFR harmonic oscillator and the NC relativistic particle developed in [96], we have shown that in both NC formalisms (a DFR algebra and a non-DFR algebra, respectively) we have a kind of duality $\theta_{const.} \longrightarrow \theta_{variable}$ which can be also represented by $\pi = 0 \longrightarrow \pi \neq 0$ or (non-Lorentz invariance) $\longrightarrow$ (Lorentz invariance) maps. The conclusion is that there is no difference between DFR and DFR*, and consequently the DFR formalism has the conjugated pairs (x, p) and $(\theta,\ \pi)$. We believe that this result complements the DFR literature. In this way, we also have constructed also the scalar field QFT and we have calculated the operatorial commutation relations with the $(x, p,\ \theta,\ \pi)$ phase-space.

The NC relativistic particle shows, besides the $\theta_{const.} \longrightarrow \theta_{variable}$ duality, another interesting result. Since the equations of motion have shown that for $\theta = const.$ we have the multiplier $\lambda_\theta = 0$ and this value zeroes the NC acceleration, the velocity is not constant since it has a parameter that is time dependent. In [96] the author has obtained this last result also, but since he does not have the value of the acceleration, it was not possible to see how interesting this result is. Besides, we have calculated here that $\dot{e} \neq 0$, which confirms that, following the equations of motion, the velocity $\dot{x}$ is not constant. It was important to compute $\dot{e}$ because although it is defined as $e = e(\tau)$ its calculation could result as zero, which would show a paradox. But it did not happen.

As a perspective we can analyze other $\theta_{variable}$ algebras different from DFR (of course) to verify if the behavior is the same. Another possible research is to construct the fermion DFR QFT. It is an ongoing research and it will published elsewhere.

In this thesis we have briefly reviewed the proposal of the NC extension of the Minkowski spacetime in which a proper time is defined in order to connect the κ-Minkowski spacetime and the extended Minkowski spacetime. We saw that in this formalism the information of

NCy can be encoded from the κ-Minkowski spacetime into the extended spacetime.

Chiral Schwinger models possess only a semilocal form of gauge invariance. However, it is well-known that one can recover full local gauge invariance by soldering two opposite chirality chiral Schwinger models. In this case one ends up with a vector Schwinger model.

Next, we have applied the soldering formalism to three 2D models: the interacting model of NC Floreanini-Jackiw chiral bosons and gauge fields, the NC generalized chiral Schwinger model and its NC gauge invariant formulation. As a result we could fuse the massless chiral states of these theories and yield equivalent non-chiral massive models. These new bosonic models have the same generating functional of the NC chiral Schwinger fermions.

Also, we have studied the bosonization of 3D Thirring model and after that by means of soldering formalism, we fused two distinct non-invariant NC (anti)self dual models to obtain a NC gauge invariant massive 3D bosonic model which is NC Chern-Simons theory in κ-Minkowski spacetime. This new massive theory is equivalent to fermionic Thirring model in the extended Minkowski spacetime and their generating functional are the same.

NC gauge theories for an arbitrary gauge group were studied and we saw that using the enveloping algebra of a Lie algebra we can construct the NC counterpart of a commutative theory. Because of ambiguity in the SW map and the mentioned no-go theorem the particle content of a NC theory is not necessarily equal to commutative counterpart and this matter is still under intense investigation in the literature [80, 119].

The JP model is a 3D massive gauge invariant model that respects the parity was introduced and we have analyzed its gauge structure and its difficulties of quantization. The NC SU(N) counterpart of this theory was constructed such that it is invariant under the same gauge form.

The field-antifield formalism is the most powerful approach to study the gauge structure and quantization of gauge theories. This formalism has an important role in quantization of modern theories of high energy physics such as supergravity and superstring theory. In the chapter 7.1 we have presented a field-antifield treatment of JP model. In fact we used this theory as a simple toy model for studying the mentioned formalism. As it was shown, the JP model has an irreducible, closed Lie algebra. After that we suggested a non-minimal gauge-fixed action for this model which is ready for further calculations such as quantization, Slavnov-Talyor identities and anomaly studies. It is worth to mention that this model has been analyzed by a slightly different approach from antifield formalism in [120].

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Appendix A – The Moyal-Weyl product

To investigate field theories defined on spaces with NC coordinates corresponding to deformations of flat spaces, as e.g. the Euclidean plane or Minkowski space $\mathbb{M}^d$ one must replace the (commuting) coordinates of flat space by Hermitian operators x^μ (with $\mu = 0$, $1, \cdots, (d\text{-}1)$) [121]. Let us consider a canonical structure defined by the following algebra

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu} \\ [\theta^{\mu\nu}, \hat{x}^\rho] &= 0, \end{aligned} \tag{A.1}$$

where simplest case is when the $\theta^{\mu\nu}$ matrix is constant, which means that we have only the first equation of (A.1). Furthermore, $\theta^{\mu\nu}$ is real and antisymmetric. In natural units, where $\hbar = c = 1$, it can be seen easily from (A.1) that it has squared mass dimension, i.e., $[\theta] = -2$.

In order to construct the perturbative field theory formulation, it is more convenient to use fields $\Phi(x)$ (which are functions of ordinary commuting coordinates) instead of operator valued objects like $\hat{\Phi}(\hat{x})$. To be able to consider such fields, concerning the properties (A.1), one must redefine the multiplication law of functional (field) space. One therefore defines the linear map $\hat{f}(\hat{x}) \longmapsto S[\hat{f}](x)$, called the "symbol" of the operator $\hat{f}$, and it can then represent the original operator multiplication in terms of the so-called star products which symbols is

$$\hat{f}\hat{g} = S^{-1}\left[S[\hat{f}] \star S[\hat{g}]\right] , \tag{A.2}$$

see for example references [122, 86]. By using the Weyl-ordered symbol (which corresponds to the Weyl-ordering prescription of the operators) one can arrive at the following definitions (with $S[\hat{f}](x) = \Phi(x)$)

$$\begin{aligned} \hat{\Phi}(\hat{x}) &\longleftrightarrow \hat{\Phi}(x), \\ \hat{\Phi}(\hat{x}) &= \int \frac{d^dk}{(2\pi)^d}\ \tilde{\hat{\Phi}}(k) e^{ik\hat{x}}, \\ \tilde{\Phi}(k) &= \int d^dx\ \Phi(x) e^{-ikx} , \end{aligned} \tag{A.3}$$

where k and x are real variables. For any two arbitrary scalar fields $\hat{\Phi}_1$ and $\hat{\Phi}_2$ one therefore can write that[7]

$$\begin{aligned} \hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x}) &= \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \tilde{\hat{\Phi}}_1(k_1)\tilde{\hat{\Phi}}_2(k_2)\ e^{ik_1\hat{x}}\ e^{ik_2\hat{x}} \\ &= \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \tilde{\hat{\Phi}}_1(k_1)\tilde{\hat{\Phi}}_2(k_2)\ e^{i(k_1+k_2)\hat{x} - \frac{1}{2}[\hat{x}^\mu, \hat{x}^\nu]k_{1,\mu}k_{2,\nu}} \end{aligned} \tag{A.4}$$

[7] One has to use the Baker-Campbell-Hausdorff formula, as well as relation (A.1)

Hence one has the following Weyl-Moyal correspondence

$$\hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x}) = \hat{\Phi}_1(x) \star \hat{\Phi}_2(x), \tag{A.5}$$

where, in using relation (A.1) to replace the commutator in the exponent of (A.4), the Moyal-Weyl star product is given by

$$\Phi_1(x) \star \Phi_2(x) = \Phi_1(x)\exp\Big(\frac{i}{2}\overleftarrow{\partial}_x\theta^{\mu\nu}\overrightarrow{\partial}_y\Big)\Phi_2(y)\Big|_{x=y}. \tag{A.6}$$

This equation means that we can work in the same way as in a usual commutative space for which the multiplication operation is modified by the star product (A.6).

Using integration by part and antisymmetric property of the NCy parameter, $\theta^{ij} = -\theta^{ji}$, we can find an useful relation

$$\int d^n x\, f(x) \star g(x) = \int d^n x\, g(x) \star f(x) = \int d^n x\, f(x)g(x). \tag{A.7}$$

The Moyal-Weyl star product in its general form is defined by

$$f_1(x_1) \star \ldots \star f_n(x_n) = \prod_{1\leq a<b\leq n} exp(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_a^i}\frac{\partial}{\partial x_b^j})f_1(x_1)\ldots f_n(x_n) \tag{A.8}$$

For instance, the star product of three functions at the first order of θ is given by

$$\begin{aligned} f(x) \star g(x) \star h(x) &= f(x)g(x)h(x) + \tfrac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x).h(x) \\ &+ \tfrac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)g(x)\partial_\nu h(x) + \tfrac{i}{2}\theta^{\mu\nu} f(x)\partial_\mu g(x)\partial_\nu h(x) \end{aligned} \tag{A.9}$$

and for the four functions we obtain

$$\begin{aligned} f(x) \star g(x) \star h(x) \star k(x) &= f(x)g(x)h(x)k(x) + \tfrac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x).h(x).k(x) \\ &+ \tfrac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)g(x)\partial_\nu h(x).k(x) + \tfrac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)g(x).h(x)\partial_\nu k(x) \\ &+ \tfrac{i}{2}\theta^{\mu\nu} f(x)\partial_\mu g(x)\partial_\nu h(x).k(x) + \tfrac{i}{2}\theta^{\mu\nu} f(x)\partial_\mu g(x)h(x)\partial_\nu k(x) \\ &+ \tfrac{i}{2}\theta^{\mu\nu} f(x)g(x)\partial_\mu h(x)\partial_\nu k(x) \end{aligned} \tag{A.10}$$

For the ordinary commuting coordinates, the Moyai-Weyl star product implies that[8]

$$\begin{aligned} [x^\mu, x^\nu]_\star &= i\theta^{\mu\nu} \\ [\theta^{\mu\nu}, x^\rho]_\star &= 0\ . \end{aligned} \tag{A.11}$$

8 The Weyl bracket is defined as $[A, B]_\star = A \star B - B \star A$

At this point one has to mention that the commutation relations (A.1) between the coordinates explicitly break Lorentz invariance because of the fact that we assumed that θ is a constant matrix [86].

Some other possibilities for a non-constant θ are, for example, $\theta^{\mu\nu} = C^{\mu\nu}_{\rho} x^\rho$ (Lie algebra) or $\theta^{\mu\nu} = R^{\mu\nu}_{\rho\sigma} x^\rho x^\sigma$ (quantum space structure) - one can see for instance reference [122, 86] for a detailed discussion about these two approaches.

Another solution of this problem leads us to the NC formulation of the spacetime used here which was formulated by Doplicher, Fredenhagen and Roberts (DFR) [69], which is based in general relativity and quantum mechanics arguments. This formalism recovers Lorentz invariance through the promotion of $\theta^{\mu\nu}$ to be a standard coordinate operator of this extra dimensional system. Of course, being the coordinate, the algebra turns out to be, together with Eq. (A.1)

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \quad ; \quad \left[\hat{x}^\mu, \hat{\theta}^{\mu\nu}\right] = 0 \quad ; \quad [\hat{p}_\mu, \hat{p}_\nu] = \left[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\rho\lambda}\right] = 0 \tag{A.12}$$

which completes the basic DFR algebra.

Appendix B – A brief review of non-Abelian gauge theories

In this appendix we are going to review some basic concepts about special unitary $su(N)$ Lie algebras[9], i. e., the algebras associated with the symmetry groups that are beneath the symmetries of the standard model of particle physics. Lie algebras are closely related to Lie groups which are groups that are also smooth manifolds, with the property that the group operations of multiplication and inversion are smooth maps. Any Lie group gives rise to a Lie algebra. Conversely, to any finite-dimensional Lie algebra over real or complex numbers, there is a corresponding connected Lie group unique up to covering. Generally physicist prefer to study Lie alberas which are linear objects instead of study Lie groups, which are geometric objects. The generators (bases of the vector space) of a general Lie algebra obey a non-associative multiplication called *Lie Bracket* [10]

$$\left[T^a \, , \, T^b\right] = i f^{abc} \, T^c \tag{B.1}$$

where the constants f^{abc} are called *structure constants* of the groups and are unique characteristics of a Lie group. The indices run from 1 to the order of the algebra, i. e. the number of generators of the algebra. The number of generators of $su(N)$ is equal to $N^2 - 1$, so the order of the group is $N^2 - 1$. A basis always can be chosen in a form that the structure constants are completely antisymmetric. One can check that any three generators A, B and C, satisfy the *Jacobi identity*

$$[[A, B] \, , C] + [[B, C] \, , A] + [[C, A] \, , B] = 0 \tag{B.2}$$

This leads to the following identity for the structure constants

$$f^{abe} f^{ecd} + f^{bce} f^{ead} + f^{cae} f^{ebd} = 0 \tag{B.3}$$

In matrix representation of SU(N), the generators are traceless but the trace of the product of two generators is in general nonzero. It is always possible to choose a basis such that

$$Tr(T^a \, T^b) = C(R) \delta^{ab} \tag{B.4}$$

where the coefficient $C(R)$ depends on the representation of the generators T^a and their normalization. However, once the this coefficient is fixed for one particular representation, the values of $C(R)$ for other representations are automatically determined.

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[9] In the literature the symbol $SU(N)$ is reserved for the special unitary Lie group and $su(N)$ for its associated Lie algebra.

[10] Generally, mathematicians define the Lie algebra in a slightly different way. In physics we are mainly interested in the Hermitian operators, for this reason we defined the Lie algebra with the complex variable "i". In this way the generators are Hermitian objects. But if we drop out this complex variable and insist on maintaining the structure constant to be real valued, the generators will be anti-Hermitian objects.

Here we will be concerned only with two specific representations of $su(N)$: the fundamental and the adjoint representations. The dimension of the fundamental representation is N, so the generators are $N \times N$ matrices and it is the smallest irreducible representation. There is also an "anti-fundamental" representation which is a complex conjugation of the fundamental representation. For $N = 2$ these two representations are actually equivalent because they are related by a unitary transformation. But for $N > 2$ the fundamental and anti-fundamental representations are not equivalent. This is the case of $SU(3)$ as the gauge group of QCD that leads to quarks and anti-quarks representations.

Usually we normalize the generators of the fundamental representation as

$$Tr(T_F^a T_F^b) = \frac{\delta^{ab}}{2} \tag{B.5}$$

and, as mentioned before, this normalization automatically sets all other coefficients of representations.

For $su(2)$ algebra, the structure constants are given by the Levi-Civita symbol, $f^{abc} = \epsilon^{abc}$. The generators of the fundamental representation of this algebra are taken to be one half of the Pauli matrices:

$$T_F^a = \frac{\sigma^a}{2} \tag{B.6}$$

where σ^a are just the usual Pauli matrices. We can see that these generators satisfy the condition (B.5).

The non-zero structure constants for the $su(3)$ algebra are

$$f^{123} = 1 \ , \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2} \ , \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2} \tag{B.7}$$

and the generators of the fundamental representation are given by the one half of the Gell-Mann matrices:

$$T_F^a = \frac{\lambda^a}{2} \tag{B.8}$$

where the Gell-Mann matrices are

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{B.9}$$

These matrices, by definition, are traceless, Hermitian, and obey the extra relation (B.5). These properties were chosen by Gell-Mann because they then generalize the Pauli matrices for $SU(2)$ group.

In the standard model, all the fermions as well as the Higgs fields transform in the fundamental and anti-fundamental representations of the gauge groups.

The adjoint representation also has a particular importance because the gauge field strengths transform in this representation. The adjoint representation is defined by

$$(T^a_{AD})^{bc} = -if^{abc}. \tag{B.10}$$

On the left hand side, the index a determines the specific generator which we are dealing with, whereas the bc label the specific entry of that generator. Clearly the dimension of this representation is $N^2 - 1$, the same value as the number of the generators. Using the Jacobi identity we can show that these matrices indeed satisfy the basic definition of the Lie groups. By assuming the specific normalization (B.5) for the fundamental representation, the adjoint representation obeys

$$Tr(T^a_{AD}\, T^b_{AD}) = N\delta^{ab}. \tag{B.11}$$

Now let us consider a set of N fermionic/bosonic fields ψ_i that transform in the fundamental representation of $sn(N)$ algebra. It means that they can be written as a column vector ψ. Under a gauge transformation these fields transform as

$$\begin{aligned}\psi \to \psi' &= \exp\left[iT^a_F \Lambda^a(x)\right]\psi \\ &\equiv U\psi\end{aligned} \tag{B.12}$$

where $\Lambda^a(x)$ are a set of $N^2 - 1$ spacetime dependent gauge parameters and U is an element of group $SU(N)$.

In the rest of the appendix we will suppress the spacetime dependence of Λ to alleviate the notation. We also define

$$T^a_F\, \Lambda^a \equiv \Lambda. \tag{B.13}$$

We have to keep in mind that Λ is an $N \times N$ matrix containing a set of $N^2 - 1$ parameters. The gauge covariant derivative on this multiplet is defined by

$$D_\mu \psi \equiv (\partial_\mu + A^a_F\, T^a_F)\psi \tag{B.14}$$

and, as it can be seen from this definition there is the same number of gauge fields as the number of generators of the group, i. e., $N^2 - 1$. To simplify the notation we also define

$$A^a_\mu\, T^a_F \equiv A_\mu \tag{B.15}$$

which we shall call A_μ as the gauge field but indeed it is a matrix as Λ [11].The transformation of the gauge field is given by

$$A_\mu \to A'_\mu = U A_\mu U^\dagger + i(\partial_\mu)U^\dagger. \tag{B.16}$$

[11] In the standard model the notation $A^a_\mu \equiv W^a_\mu$ for the $SU(2)_L$ gauge fields and $A^a_\mu \equiv G^a_\mu$ for the glouns

By adopting this definition the covariant derivative $D_\mu\psi$ transforms in the same way as ψ itself

$$D_\mu\psi \to (D_\mu\psi)' = UD_\mu\psi \tag{B.17}$$

As an example let us take a look at the transformation of Dirac spinors Ψ_i. When these fields transform in the fundamental representation, a gauge invariant Lagrangian can be obtained simply by replacing ∂_μ by the gauge covariant derivative in the Dirac Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{B.18}$$

where, again, Ψ is seen as a column vector containing the N Dirac spinors transforming in the fundamental representation of a group[12].

It is noteworthy to mention that there is much freedom in the choice of signs in the definitions (B.14) and (B.16), the only restriction is that these definitions must result transformation property (B.17).

The field strength corresponding to a gauge field is

$$\begin{aligned} F_{\mu\nu} &= [D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \end{aligned} \tag{B.19}$$

The commutator is, of course, zero in the case of an Abelian gauge theory like Maxwell electrodynamics. Since the field strength is also a Lie algebra-valued object, it may be written explicitly as

$$\begin{aligned} F_{\mu\nu} &\equiv F^a_{\mu\nu}\, T^a_F \\ &= \partial_\mu A^a_\nu\, T^a_F - \partial_\nu A^a_\mu\, T^a_F - iA^b_\mu\, A^c_\nu \left[T^b_F\,,\, T^c_F\right] \end{aligned} \tag{B.20}$$

Using the basic property of Lie bracket we have

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu\, A^c_\nu \tag{B.21}$$

Using Eq. (B.16) one can show that the field strength transforms as

$$\begin{aligned} F_{\mu\nu} \to F'_{\mu\nu} &= UF_{\mu\nu}U^\dagger \\ &\approx F_{\mu\nu} + i\,[\Lambda, F_{\mu\nu}] \\ &= F^a_{\mu\nu}T^a_F + i\Lambda^b\, F^c_{\mu\nu}\left[T^b_F\,,\, T^c_F\right] \\ &= F^a_{\mu\nu}T^a_F - f^{abc}\Lambda^b F^c_{\mu\nu}T^a_F \end{aligned} \tag{B.22}$$

from which we can read off the transformation of $F^a_{\mu\nu}$:

$$\left(F^a_{\mu\nu}\right)' = F^a_{\mu\nu} - f^{abc}\Lambda^b F^c_{\mu\nu} \tag{B.23}$$

[12] Replacing normal derivative by the covariant one is a common method for introducing the interaction of a particle with a gauge field

Using the antisymmetry of the structure constants and the definition of the adjoint representation we can rewrite the above equation as

$$\left(F^a_{\mu\nu}\right)' = F^a_{\mu\nu} + i\left(T^b_{AD}\right)^{bc}\Lambda^b F^c_{\mu\nu} \tag{B.24}$$

which is the infinitesimal limit of

$$\left(F^a_{\mu\nu}\right)' = \exp\left(i\,T^b_{AD}\Lambda^b\right)F_{\mu\nu} \tag{B.25}$$

This shows that the field strength transforms in the adjoint representation. Note that in the above equation, $F_{\mu\nu}$ is a column vector with N^2-1 components, in contrast with the $F_{\mu\nu}$ in the Eq. (B.22) which is an $N\times N$ matrix.

Unlike the Abelian case, $F^{\mu\nu}F_{\mu\nu}$ of a non-Abelian gauge theory is not gauge invariant and it is a matrix. So we can not use it as the kinetic term of a Lagrangian. On the other hand, the trace of this quantity is gauge invariant and the kinetic term can be taken in the following form

$$\begin{aligned}\mathcal{L}_{kin} &= -\frac{1}{2}Tr\left(F^{\mu\nu}F_{\mu\nu}\right)\\ &= -\frac{1}{2}F^{a\mu\nu}F^b_{\mu\nu}Tr\left(T^a_F\,T^b_F\right)\\ &= -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\end{aligned} \tag{B.26}$$

Appendix C – A quick glance at universal enveloping algebra

Here we would like to take a glance at the concept of universal enveloping algebra. Universal enveloping algebras are nearly as common as Lie algebras in physics, but we often take them for granted or do not even think about them. The universal enveloping algebra of a Lie algebra is the most general unital associative algebra into which the Lie algebra can be embedded.

Let us consider a Lie algebra generated by T_i, $i = 1, 2, \cdots, n$,

$$[T_i\ ,\ T_j] = i f_{ijk} T_k \tag{C.1}$$

The Lie algebra could for instance be the algebra of angular momentum operators J_i , a cornerstone of quantum mechanics. We do not usually see the Lie bracket $[T_i\ ,\ T_j]$ as a non-associative product of generators T_i , but instead as a commutator

$$[T_i\ ,\ T_j] \equiv T_i T_j - T_j T_i = i f_{ijk} T_k \tag{C.2}$$

where the associative product is $T_i T_j$. Thus, we have embedded the Lie algebra into its universal enveloping algebra that consists of the polynomials in the generators T_i modulo the commutation relations (C.2) and of the unit element $\mathbf{1}$. The basis of the universal enveloping algebra can be chosen such that it consists of $\mathbf{1}$ and of the fully symmetrized products of the generators

$$T_{(i_1} T_{i_2} \cdots T_{i_n)} \quad , \quad n \in \mathbb{N} \tag{C.3}$$

Since the universal enveloping of a Lie algebra fully captures the structure of the Lie algebra, the representation of the common generators are identical for the two algebras. In the universal enveloping of a Lie algebra we can define such polynomial operators as the quadratic Casimir operators, which can be used to classify the representations of the corresponding Lie algebra.

Every Lie algebra has a universal enveloping algebra, which is uniquely determined up to a unique algebra isomorphism by the Lie algebra. This property of "universality" is the reason why enveloping algebras of Lie algebras are called universal. The associativity property of universal enveloping algebras enables the introduction of interesting additional structures and that is what makes universal enveloping algebras so useful for us.