Lucas Tavares Cardoso

# Towards Renormalizability of String-Localized Massive Quantum Electrodynamics

Juiz de Fora - MG, Brasil

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Tese apresentada ao Programa de Pós Graduação em Física do Instituto de Ciências Exatas da Universidade Federal de Juiz de Fora como requisito parcial a obtenção do título de Doutor em Física.

Universidade Federal de Juiz de Fora – UFJF

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Supervisor: Jens Karl Heinz Mund

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Aprovada em:

BANCA EXAMINADORA

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*I dedicate this thesis to my loving wife Ninfa, my two gracious children Alice and Pedro Lucas, and my parents Cláudia and Paulo. My world would lose its entire meaning without you.*

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*"If we will disbelieve everything, because we cannot certainly know all things, we shall do muchwhat as wisely as he who would not use his legs, but sit still and perish, because he had no wings to fly."* **John Locke**

# Resumo

A construção de campos com localização em cordas foi realizada rigorosamente há pouco mais de uma década. Nesta abordagem, os campos são operadores em algum espaço de Hilbert, e portanto não há graus de liberdade não-físicos tais como "ghosts". Além de permitir a construção de campos carregados inteiramente num espaço espaço de Hilbert, os campos com localização em cordas exibem um bom comportamento no regime ultravioleta e, entre outras características, são possíveis candidatos para descreverem consistentemente a matéria escura. No intuito de obter uma prova da renormalizabilidade em modelos perturbativos no esquema de Epstein-Glaser com campos quânticos localizados em cordas, é necessário evidenciar a liberdade que se tem ao definir produtos temporalmente ordenados do Lagrangeano de interação. Este trabalho proporciona um primeiro passo significativo nesta direção. O problema básico é a presença de um conjunto aberto de $n$-uplas de cordas que não podem ser cronologicamente ordenadas. Nós resolvemos este problema ao mostrar que quase todas (i.e. exceto num subconjunto de medida nula) tais configurações de cordas podem ser dissecadas num número finito de pedaços, os quais por sua vez podem ser cronologicamente ordenados. Com isso, tem-se que o produto temporalmente ordenado de fatores lineares de campos está fixo fora de um conjunto de medida nula de configurações de cordas. A construção do ordenamento temporal geométrico de cordas é feita de modo a servir para o estudo da renormalizabilidade de quaisquer teorias quânticas de campos com localização em cordas.
**Palavras-chaves**: Campos quânticos. Localização em cordas. Corte de uma corda. Produtos temporalmente ordenados. Esquema de Epstein-Glaser. Renormalização. Renormalização perturbativa.

# Abstract

The construction of string-localized fields was rigorously accomplished a little over a decade ago. In this approach, the fields are operators in some Hilbert space, and therefore there are no unphysical degrees of freedom such as ghosts. In addition to allowing the construction of charged fields entirely in a Hilbert space, the string-localized fields exhibit, in general, a good behavior in the ultraviolet regime and, among other features, the class (representation) of string-localized fields with $m = 0$ and $s = \infty$ are possible candidates to consistently describe dark matter. In order to obtain a proof of renormalizability of perturbative models in the Epstein–Glaser scheme with string-localized quantum fields, one needs to know what freedom one has to define time-ordered products of the interaction Lagrangian. This work provides a first significant step in that direction. The basic issue is the presence of an open set of $n$-tuples of strings which cannot be chronologically ordered. We resolve it by showing that almost all (i.e. outside a null set) such string configurations can be dissected into finitely many pieces which can indeed be chronologically ordered. This fixes the time-ordered products of linear field factors outside a nullset of string configurations. The construction of the geometric time ordering of strings is realized in such a way that it will serve to study the renormalizability of any quantum field theories with string-localized fields.
**Keywords**: Quantum Fields. String-localization. Chopping of a string. Time-ordered products. Epstein-Glaser scheme. Renormalization. Perturbative renormalization.

# Contents

# Introduction

The three pillars of relativistic quantum field theory (QFT) are positivity of states, positivity of the energy and locality of observables (or Einstein causality). Any attempt to reconcile them leads to the well-known singular behaviour of quantum fields at short distances (UV singularities) [1], which becomes worse with increasing spin. This rules out the direct construction of interacting models for particles with spin/helicity $s \geq 1$ in a frame which incorporates the three principles from the beginning.

The usual way out is gauge theory (GT), where one relaxes the principle of positivity of states in a first step, and divides out the unphysical degrees of freedom (negative norm states and ghost fields) at the end of the construction. This approach has been extremely successful and is the basis of the Standard Model of elementary particle physics. However, it has some shortcomings: The intermediate use of unphysical degrees of freedom does not comply well with Ockham's razor; The approach does not provide a direct construction of charge carrying physical fields; It excludes an energy-momentum tensor for massless higher helicity particles [2]; Finally, many features of models must be put in by hand instead of being explained, like for example the shape of the Higgs potential, and chirality of the weak interactions.

There is an alternative, relatively recent but conservative approach [3–6], which keeps positivity of states and instead relaxes the localization properties of (unobservable) quantum fields: These fields are not point-local, but instead are localized on Mandelstam strings extending to space-like infinity [3, 7]. Such a string, not to be confused with the strings of string theory, is a ray emanating from an event $x$ in Minkowski space in a space-like[1] direction $e$,

$$S_{x,e} \doteq x + \mathbb{R}_0^+ e. \tag{1}$$

Our quantum fields are operator-valued distributions $\varphi(x, e)$, where $x$ is in Minkowski space and $e$ is in the submanifold of space-like directions

$$H := \{e \in \mathbb{R}^4 : \; e \cdot e = -1\}. \tag{2}$$

The field $\varphi(x, e)$ is localized on the string $S_{x,e}$ in the sense of compatibility of quantum observables: If the strings $S_{x,e}$ and $S_{x',e'}$ are space-like separated,[2] then

$$[\varphi(x, e), \varphi(x', e')] = 0. \tag{3}$$

1 The choice of *space*-like strings is motivated by the known fact that in every massive model charge-carrying field operators are localizable in space-like cones [8]. It seems, however, that our constructions go through also for *light*-like strings, replacing $H$ by the forward light cone.

2 Indeed, the distributional character of the fields requires that $S_{x',e''}$ be space-like separated for all $e''$ in an open neighborhood of $e'$ [3].

It has been shown [8] that in the massive case this is the worst possible "non-locality" for unobservable fields which is consistent with the three mentioned principles (in particular with locality of the observables), and that this weak type of localization still permits the construction of scattering states. Free string-localized fields for any spin with good UV behaviour have been constructed in a Hilbert space without ghosts. Among these are string-localized fields which differ from their bad-behaved point-localized counterparts by a gradient [4, 6, 9]. They allow for the construction of string-localized energy-momentum tensors for any helicity [9, 10], evading the Weinberg-Witten theorem [2]. In the (perturbative) construction of interacting models, one uses an interaction Lagrangean which differs from a point-localized counterpart by a divergence. Then the classical action is the same for both Lagrangeans. (This is analogous to gauge theory, where two Lagrangeans in different gauges yield the same action.) The requirement that this equivalence survives at the quantum level leads to renormalization conditions which we call *string independence (SI)* conditions, reminiscent of the Ward identities in gauge theory. They are quite restrictive: In particular, they imply features like chirality of weak interactions [11] , the shape of the Higgs potential [12] and the Lie-algebra structure in models with self-interacting vector bosons [5]. It is not clear at the moment if this approach leads to the same models as the gauge theoretic one.

A proof of renormalizability at all orders in this approach is missing up to date. The present work is meant as a first step in this direction. We aim at the perturbative construction of interacting models within the Epstein-Glaser scheme [13]. This approach is based on the Dyson series expansion of the $S$-matrix in terms of time-ordered products of the interaction Lagrangean, which is a Wick polynomial in the free fields. In the case of point-localized fields, renormalizability enters as follows. The time-ordered products of $n$ Wick monomials $W_i$ are basically characterized by symmetry and the factorization property, namely

$$TW_1(x_1)\cdots W_n(x_n) = TW_1(x_1)\cdots W_k(x_k)\; TW_{k+1}(x_{k+1})\cdots W_n(x_n) \tag{4}$$

whenever each event in $\{x_1,\ldots,x_k\}$ is "later" than each event in $\{x_{k+1},\ldots,x_n\}$. (We say that $x$ is later than $y$ if there is a reference frame such that $x^0 > y^0$.) Indeed, these properties fix the $T$ products outside the origin by translation invariance and induction, (see [13, 14]). In this $x$-space approach, the "UV problem" of divergencies consists in the extension across the origin, which is not unique: At every order $n$ one has a certain number of free parameters. If the short distance scaling dimension of the interaction Lagrangean is not larger than 4, then this number does not increase with the order, and one can fix all free parameters by a finite set of normalization conditions: The model is renormalizable [13].

In the present work, we initiate the corresponding discussion for string-localized quantum fields $\varphi(x,e)$ by considering time-ordered products of *linear* fields $T\varphi(x_1,e_1)\cdots\varphi(x_n,e_n)$. (The case of Wick monomials of order $> 1$ is left for a future publication.) These are required to be symmetric and to satisfy the factorization property, namely Eq. (4) must hold, with $W_i(x_i)$ replaced by $\varphi(x_i, e_i)$, whenever each of the first $k$ strings is later than each of the last $n-k$ strings.

The basic problem, already present at order 2, is that two strings generically are not comparable in the sense of time-ordering. In fact, there is an open set of pairs $(x, e), (x', e')$ corresponding to strings which are not comparable, (see Lemma 2.1.1). Thus the $T$ product of two fields is undefined on an open set, which leaves an infinity of possible definitions instead of finitely many parameters already at second order, jeopardizing renormalizability. For three and more strings, the problem becomes worse (see Fig. 1 for a typical example). To overcome this problem, we prove first that $n$ strings which do not touch each other can

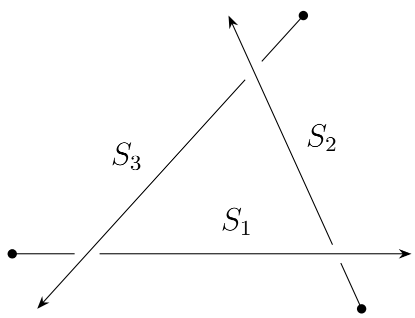


**Figure 1** – Three strings, for which no string is later than all the others (in three-dimensional space-time – the time arrow points out of the plane).

be chopped up into finitely many pieces which are mutually comparable. This is our main result. It is shown first for $n = 2$ in a constructive way (Prop. 2.2.1), and then for $n > 2$ with a non-constructive proof (Prop. 2.2.4).

We then proceed to show how this purely geometric result fixes the time-ordered products $T\varphi(x_1, e_1) \cdots \varphi(x_n, e_n)$ outside the meager set $\Delta_n$ of strings that touch each other. In particular, they turn out to satisfy Wick's expansion. Again, this is first shown for $n = 2$ (Prop. 3.2.1), where the product $T\varphi\varphi$ is fixed by its vacuum expectation value (the Feynman propagator), and then for $n > 2$ (Prop. 3.2.2).

In the extension of the $T$ products across $\Delta_n$, the scaling degrees [15] of the Feynman propagator with respect to the various submanifolds of $\Delta_2$ have to be compared with the respective co-dimensions. We give an example in Appendix B.2, but leave the general discussion open for future investigations.

We close the introduction with some further details. Our fields are covariant under a unitary representation $U$ of the proper orthochronous Poincaré group:

$$U(a, \Lambda)\, \varphi(x, e)\, U(a, \Lambda)^{-1} = \varphi(a + \Lambda x, \Lambda e), \tag{5}$$

where $a \in \mathbb{R}^4$ is a translation and $\Lambda$ is a Lorentz transformation. (This is the scalar case, which we consider here for sake of notational convenience. The fields may have vector or tensor indices which also transform.)The irreducible sub-representations of $U$ correspond to the particle types described by $\varphi$.[3] We consider here only the case of Bosons, and we exclude explicitely the case of Wigner's massless "infinite spin" particles [16]. It has been shown in [7] that then our string-localized free massive field $\varphi(x,e)$ is of the form

$$\varphi(x,e) = \int_0^\infty ds\, u(s)\varphi^p(x+se), \tag{6}$$

where $\varphi^p$ is some point-localized free field, and $u$ is some real-valued function with support in the positive reals.

Of course, one might define the time-ordered product $T\varphi(x,e)\varphi(x',e')$ by first taking the point-local one and then integrating, as in Eq. (B.12). (Our Prop.s 3.2.1 and 3.2.2 may be obtained this way.) However, when it comes to renormalization (or extension), this procedure misses the central point of our approach: The point-local Feynman propagator for higher spin fields (or derivatives of scalar fields) is not unique due to its bad UV behaviour, and leaves the freedom of adding delta function renormalizations. This freedom is not undone by the subsequent intregrations. On the other hand, the UV behaviour of $\varphi$ is better than that of the point-local field $\varphi^p$ just due to the integration, and therefore in general the $T$ product has less freedom. We give an example in Appendix B.2. We conclude that it is worthwile to take the string-localized $\varphi$ serious as the basic building block (and not to overburden the $T$ product by continuity assumptions permitting exchange of integration and time ordering).

This thesis is organized as follows. Chapter 1 is dedicated to mathematical and physical preliminaries, which contains two sections: the first concerning the geometry of Minkowski space-time and the second concerning some basic distribution theory. Chapter 2 is concerned with a thorough study of geometric time-ordering of strings in space-time: We define the time-ordering prescription for strings, *i.e.*, the "later than" relation, and prove our main geometrical result on the chopping of strings. In Chapter 3, we start with a section treating superficially the case of time-ordered point-like fields and study the renormalizability of a scalar field theory with interaction lagrangean $L(x) =: \varphi(x)^m :$ for illustrative and motivational purposes, evidentiating the stragety to be followed also in the string-like case with the necessary modifications. In the subsequent section, the axioms for the time-ordered product of string-localized fields are stated, and we show that (in the case of linear factors) it is fixed outside the set $\Delta_n$ and satisfies Wick's expansion. In Section 3.2.3, we comment on the problem arising in extending the present results to Wick monomials of order $> 1$. This work has already produced two papers, one related to section 3.1, which has already been published [17], and the other related to chapters 2 and 3(with the exception of section 3.1) which is currently being reviewed [18].

[3] Such fields exist for any spin/helicity [6, 7].

# 1 Mathematical and Physical Preliminaries

## 1.1 Geometric Structure of $\mathcal{M}$

In this section we review some fundamental geometrical properties of Minkowski space[1].

**Definition 1.1.1.** *A* ***vector space*** $\mathcal{V}$ *over a field* $\mathbb{F}$ *is a set together with two laws of composition:*

- *addition:* $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$,
- *scalar multiplication:* $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$,

*and satisfying the following axioms:*

*(i)* *Addition makes* $\mathcal{V}$ *into a commutative group* $\langle \mathcal{V}, + \rangle$.

*(ii)* *Scalar multiplication is associative with respect to multiplication in* $\mathbb{F}$*:*

$$(ab)v = a(bv), \quad \forall a, b \in \mathbb{F} \quad \textit{and} \quad \forall v \in \mathcal{V}$$

*(iii)* *The element* $1_{\mathbb{F}}$ *acts as identity:* $1_{\mathbb{F}} v = v$, $\forall v \in \mathcal{V}$.

*(iv)* *Two distributive laws hold:*

$$(a+b)v = av + bv, \quad \textit{and} \quad a(v+w) = av + aw, \quad \forall a, b \in \mathbb{F} \quad \textit{and} \quad \forall v, w \in \mathcal{V}$$

**Definition 1.1.2.** *A* ***bilinear form*** *on a vector space* $\mathcal{V}$ *is a map* $g : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ *that is linear in each variable. Also, we define a bilinear form to be:*

- ***symmetric*** *if* $g(v, w) = g(w, v) \quad \forall v, w \in \mathcal{V}$;
- ***positive [negative] definite*** *if* $g(v, v) > 0 \; [< 0] \quad \forall v \in \mathcal{V} \setminus \{0\}$;
- ***positive [negative] semidefinite*** *if* $g(v, v) \geq 0 \; [\leq 0] \quad \forall v \in \mathcal{V} \setminus \{0\}$;
- ***nondegenerate*** *if* $g(v, w) = 0 \quad \forall w \in \mathcal{V}$ *implies* $v = 0$;

[1] For wonderful studies on the geometry of Minkowski space see [19, 20]

**Remark 1.1.3.** *Recall that an inner product on a vector space $\mathcal{V}$ (usually denoted by $\langle , \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$) is a positive-definite symmetric bilinear form. The Euclidean space, which is one of the most important types of inner product spaces, is endowed with the correct kind of structure to serve as a mathematical model for many physical systems. However, an inner product space is not structurely adequate to accomodate the Special Theory of Relativity, since here, the "distance" between two events is no longer positive-definite. For this purpose, the following defined structure was devised.*

**Definition 1.1.4.** ***Minkowski space***[2] *is a 4-dimensional real vector space $\mathcal{M}$ on which is defined a nondegenerate, symmetric, bilinear form $g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ given by $g(v, w) = v_0 w_0 - v_1 w_1 - v_2 w_2 - v_3 w_3 =: v \cdot w$ for every $v, w \in \mathcal{M}$, where $v = (v_0, v_1, v_2, v_3)$ and $w = (w_0, w_1, w_2, w_3)$. The elements in $\mathcal{M}$ are called* ***events*** *and $g$ is referred to as the* ***Lorentz inner product*** *on $\mathcal{M}$.*

**Remark 1.1.5.** *Notice here, that the Lorentz inner product (from now on, we may call it just pruduct) can be any real number. Also, the* ***quadratic form*** *associated with the inner product $g$ on $\mathcal{M}$ is the map $q : \mathcal{M} \to \mathbb{R}$ defined by $q(v) = g(v, v) =: v^2$. Two vectors $x, y \in \mathcal{M}$ are said to be* ***g-othogonal*** *or just* ***orthogonal*** *if $g(x, y) = 0$. Given a subset $\mathcal{A} \subseteq \mathcal{M}$,its orthogonal complement $\mathcal{A}^{\perp}$ in $\mathcal{M}$ is thus defined by $\mathcal{A}^{\perp} = \{v \in \mathcal{M} : \ g(v, w) = 0 \quad \forall w \in \mathcal{M}\}$. A* ***unit vector*** *$v$ is a vector for which either $q(v) = 1$ or $q(v) = -1$. A basis $\{e_0, e_1, e_2, e_3\}$ for $\mathcal{M}$ consisting of unitary orthogonal vectors is called an* ***orthonormal basis*** *for $\mathcal{M}$. An orthonormal basis $\{e_0, e_1, e_2, e_3\}$ for $\mathcal{M}$, in a way, coordinatizes every event and can be identified with a frame of reference. Thus, if $x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3$, we regard the coordinates $(x^0, x^1, x^2, x^3)$ of $x$ relative to the basis $\{e_\alpha\}$ as the time $(x^0)$ and spatial $(x^1, x^2, x^3)$ coordinates supplied the event $x$ by the observer who presides over this frame of reference. Also, the inner product between two vectors $v, w \in \mathcal{M}$ may be written, using Einstein's summation convention, as*

$$v \cdot w = \eta_{\mu\nu} v^\mu w^\nu, \tag{1.1}$$

*where $\eta_{\mu\nu}$ (or $\eta^{\mu\nu}$) are the components of the following $4 \times 4$ matrix*

$$\eta = (\eta_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{1.2}$$

*that is, $\eta_{\mu\nu} = \eta^{\mu\nu} = 0$ if $\mu \neq \nu$, $\eta_{00} = \eta^{00} = 1$, and $\eta_{ii} = \eta^{ii} = -1$ if $i = 1, 2, 3$.*

[2] It should be pointed out that due to its vector space structure, Minkowski space can be canonically identified with its own tangent space $T_p\mathcal{M}$ at the origin [21, p. 15]. Therefore, we shall make no distinction between vectors in $T_p\mathcal{M}$ and vectors in $\mathcal{M}$.

**Definition 1.1.6.** *Let $\mathcal{M}$ be a Minkowski vector space. A **Lorentz transformation** on $\mathcal{M}$ is an isomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ which satisfies the following condition*

$$g(\varphi(v), \varphi(w)) = g(v, w) \quad \forall v, w \in \mathcal{M}. \tag{1.3}$$

Thus, a Lorentz transformation "preserves all the structure of Minkowski space." Furthermore, let

$$\mathcal{L} := \{\varphi : \mathcal{M} \to \mathcal{M} : \text{such that } g(\varphi(v), \varphi(w)) = g(v, w) \quad \forall v, w \in \mathcal{M}\}, \tag{1.4}$$

then

**Theorem 1.1.7.** *The set $\mathcal{L}$ of all isomorphisms from $\mathcal{M}$ to $\mathcal{M}$ forms a group under composition, called the **Lorentz group**.*

**Remark 1.1.8.** *The postulates of the Theory of Special Relativity (TSR), imply that given two events $P = (P_0, P_1, P_2, P_3)$ and $Q = (Q_0, Q_1, Q_2, Q_3)$ in space-time, the interval $\Delta S$ between them, defined by $\Delta S^2 \doteq (P_0 - Q_0)^2 - (P_1 - Q_1)^2 - (P_2 - Q_2)^2 - (P_3 - Q_3)^2$, is invariant under a change of inertial frames of reference. Therefore, rigorously speaking, the space-time of TSR is, in a sense, more accurately, structurized mathematically as an affine space (or, linear manifold), rather than a vector space, since physically there is no preferred event in space-time. Let us denote **Minkowski space-time** by $\mathbb{M}$. The relation between $\mathbb{M}$ and $\mathcal{M}$ can be characterized by the map*

$$vec : \mathbb{M} \times \mathbb{M} \to \mathcal{M}, \tag{1.5}$$

*given by*

$$vec(P, Q) := \overrightarrow{PQ} \tag{1.6}$$

*and for which*

$$vec(P, Q) + vec(Q, R) = vec(P, R), \tag{1.7}$$

*where $P, Q, R \in \mathbb{M}$. It follows from (1.7) that $vec(P, P) = \overrightarrow{0}$ and $vec(P, Q) = -vec(Q, P)$. Thus, we may write the the interval $\Delta S^2 = \overrightarrow{PQ} \cdot \overrightarrow{PQ}$ and now clearly a Lorentz transformation leaves the interval invariant. However, by homegeneity of space-time, even though translated frames of reference also yield the same interval between two arbitrary events, translations do not satisfy equation (1.3)[3], that is, they are not Lorentz transformations. Furthermore, the set of all Lorentz transformations and translations are called **Poincaré transformations** which form the isometry group of space-time, called the **Poincaré group**, $\mathcal{P}$.*

[3] A translation by a fixed vector $a \in \mathcal{M}$ may be thought of as an affine transformation $T_a : \mathbb{M} \to \mathbb{M}$ given by $T_a(P) = P + a := P' \in \mathbb{M}$. Let $P$ and $Q$ be two events in $\mathbb{M}$, then $T_a(P) = P + a = P'$ and $T_a(Q) = Q + a = Q'$, and consequently $vec(T_a(P), T_a(Q)) = \overrightarrow{P'Q'} = (Q + a) - (P + a) = \overrightarrow{PQ}$, hence leaving the interval invariant.

*The Lorentz group may be seen as an isotropy subgroup of $\mathcal{P}$ which fixes the origin*[4]*. The abelian group of translations on $\mathcal{M}$, denoted by $\mathcal{R}^{1,3}$ is also a subgroup of $\mathcal{P}$. Since $\mathcal{R}^{1,3}$ is normal in $\mathcal{P}$ we may represent the latter as a semidirect product of $\mathcal{R}^{1,3}$ and $\mathcal{L}$, that is $\mathcal{P} = \mathcal{R}^{1,3} \rtimes \mathcal{L}$.*

**Remark 1.1.9.** *We may represent a Lorentz transformation as a linear transformation $\varphi_\Lambda : \mathcal{M} \to \mathcal{M}$ defined by $\varphi_\Lambda(x) := \Lambda \cdot x = x'$,* [5]*where $x \in \mathcal{M}$, and $\Lambda$ is a $4 \times 4$ matrix that leaves the Lorentz inner product invariant. After choosing a basis for $\mathcal{M}$, we may write $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda^\mu{}_\nu$ are the sixteen components of $\Lambda$ satisfying*

$$\Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta_{\mu\alpha} = \eta_{\nu\beta}, \quad \nu, \beta \in \{0,1,2,3\}, \tag{1.8}$$

*which follows from equation (1.1), and is equivalent to the relation*

$$\Lambda^T \eta \Lambda = \eta, \tag{1.9}$$

*where $T$ stands for transpose.*

**Remark 1.1.10.** *The (complete) Lorentz group, $O(1,3)$, is the group of all linear isometries of Minkowski's (pseudo-)metric. This group has four connected components, whose elements may or may not preserve either the space orientation or the time direction. The* ***orthochronous subgroup****, $O^+(1,3)$, is formed by the two preserving-time components, and the* ***proper subgroup****, $SO(1,3)$, is formed by the two components preserving space orientation, which are the elements with determinant $1$. The identity component, which preserves both time and space, is in the subgroup $SO^+(1,3)$. This subgroup is called the* ***proper, orthochronous Lorentz group*** *or the* ***restricted Lorentz group****. Only the elements of the connected component of the unit correspond to physically realizable transformations. Therefore, $\mathcal{P}^+_\uparrow := \mathcal{R}^{1,3} \rtimes SO^+(1,3)$ is considered as the relativistic invariance group. In this work, we restrict to the symmetry group $\mathcal{P}^+_\uparrow$, thus excluding parity and time reflection.*

**Remark 1.1.11.** *Similarly to Remark 1.1.9, We may represent a Poincaré transformation*[6] *by $\varphi_{(a,\Lambda)} : \mathcal{M} \to \mathcal{M}$, where $\varphi_{(a,\Lambda)}(x) = (a,\Lambda)x := a + \Lambda x$. The composition law on $\mathcal{P}^+_\uparrow$ is defined as follows: let $(a_1,\Lambda_1),(a_2,\Lambda_2) \in \mathcal{P}^+_\uparrow$, then $(a_1,\Lambda_1)\cdot(a_2,\Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)$.*

The fact that the quadratic form $q$ defined in Remark 1.1.5 can take any real number, gives rise to Minkowski's space peculiar geometrical structure. In order to view some of its idiosyncrasies we shall first need some definitions.

[4] Equivalently, we could have said that the Lorentz group is the stabilizer subgroup of $\mathcal{P}$ with respect to 0, that is $\mathcal{L} = \mathcal{P}_0 \doteq \{p \in \mathcal{P} : p(0) = 0\}$.

[5] Although the same symbol $x$ was used to denote both a vector and a $4 \times 1$ column matrix, which are obviously different objects, from the context it is going to be clear which object is being represented by $x$ and hopefully no confusion will arise whatsoever.

[6] From now on we'll refer to $\mathcal{P}^+_\uparrow$ simply as the Poincaré group, and its cooresponding elements as Poincaré transformations

**Definition 1.1.12.** *Let $q : \mathcal{M} \to \mathbb{R}$ defined by $q(v) = v^2$. We say $v$ is **time-like**, **space-like**, or **light-like** if $q(v) > 0$, $q(v) < 0$, or $q(v) = 0$, respectively.*

**Definition 1.1.13.** *The **forward lightcone** $V_+$ is the set of all time-like and future-pointing vectors, $V_+ := \{x \in \mathbb{R}^4 : x^0 > |\vec{x}|\}$, and for $z \in \mathbb{R}^4$ we denote $V_+(z) := V_+ + z$ $= \{x + z \in \mathbb{R}^4 : x^0 > |\vec{x}|\}$, where $x^0$ represents the time coordinate of $x$ and $\vec{x}$ its spatial coordinates, i.e $(x^0, \vec{x}) \doteq x$.*

*Similarly, the **backward lightcone** $V_-$ is the set of all time-like and past-pointing vectors and $V_-(z) := \{x + z \in \mathbb{R}^4 : x^0 < -|\vec{x}|\}$.*

**Remark 1.1.14.** *The boundaries of the backward and forward lightcones are given by $\partial V_-(z) := \{x + z \in \mathbb{R}^4 : x^0 = -|\vec{x}|\}$ and $\partial V_+(z) := \{x + z \in \mathbb{R}^4 : x^0 = |\vec{x}|\}$, respectively. Furthermore, the closure of the forward lightcone $V_+(z)$ is denoted $\overline{V_+(z)}$, similar for the backward lightcone.*

**Definition 1.1.15.** *For any set $R \subset \mathbb{R}^4$ the backward and forward lightcones (or causal past and future) of $R$ are defined by $V_-(R) := \bigcup_{z \in R} V_-(z)$ and $V_+(R) := \bigcup_{z \in R} V_+(z)$, respectively.*

The following theorem will be useful in chapter 2.

**Theorem 1.1.16.** *A vector $\xi \in \mathbb{R}^4$ is contained in $\overline{V_+}$ if, and only if, it satisfies $u \cdot \xi \geq 0$ for all $u \in V_+$.*

*Proof.* Let $u$ be an arbitrary vector in $V_+$ and suppose firstly that $\xi \in \overline{V_+}$. In the case where $\xi = 0$, we trivially have $u \cdot \xi = 0 \quad \forall u \in V_+$. Thus, let us consider the non-trivial case where $\xi \in \overline{V_+} \setminus \{0\}$. Then, $u^2 \doteq (u^0)^2 - \vec{u} \cdot \vec{u} > 0$ and $\xi^2 \doteq (\xi^0)^2 - \vec{\xi} \cdot \vec{\xi} \geq 0$, or equivalently

$$\begin{cases} (u^0)^2 > \vec{u} \cdot \vec{u} \\ (\xi^0)^2 \geq \vec{\xi} \cdot \vec{\xi} \end{cases}, \tag{1.10}$$

where $u^0$ is the time coordinate of the vector $u$ and $\vec{u}$ is the 3-vector composed by the spatial coordinates of $u$, and analagously for the vector $\xi$. From the two equations in (1.10) it follows that

$$\left| u^0 v^0 \right| > \sqrt{(\vec{u} \cdot \vec{u})(\vec{\xi} \cdot \vec{\xi})}. \tag{1.11}$$

Using the *Cauchy-Schwartz inequality* for the 3-vectors $\vec{u}$ and $\vec{\xi}$ given by

$$\left| \vec{u} \cdot \vec{\xi} \right| \leq \sqrt{(\vec{u} \cdot \vec{u})(\vec{\xi} \cdot \vec{\xi})}, \tag{1.12}$$

combined with equation (1.11) gives us

$$\left| u^0 \xi^0 \right| > \left| \vec{u} \cdot \vec{\xi} \right|. \tag{1.13}$$

Since $u$ and $\xi$ are both future-pointing time-like vectors, $u^0\xi^0 > 0$, from which it follows that

$$u^0\xi^0 - \vec{u} \cdot \vec{\xi} \equiv u \cdot \xi > 0, \tag{1.14}$$

and since $u$ is an arbitrary vector in $V_+$, equation (1.14) holds $\forall u \in V_+$.

Proving the other direction of the lemma is equivalent, by contraposition, to proving $\xi \notin \overline{V_+} \Rightarrow \exists u \in V_+ : u \cdot \xi < 0$. Therefore, suppose $\xi \notin \overline{V_+}$, then either $\xi \in \overline{V_-} \setminus \{0\}$, or $\xi \in \{0\}^{c}$[7].

1. $\xi \in \overline{V_-} \setminus \{0\}$. In this case, $-\xi \in \overline{V_+} \setminus \{0\}$, and by the first part of the proof, $-\xi \cdot u > 0$ and consequently $u \cdot \xi < 0$.

2. $\xi \in \{0\}^c$. In this case, $\xi^\perp$ is a time-like hyperplane (see definition A.0.3) passing through the origin. Let $u_0 \in \xi^\perp \cap V_+$ and set $u = u_0 + \epsilon\xi$, where $\epsilon$ is a small enough positive number.[8] Then,

$$u \cdot \xi = (u_0 + \epsilon\xi) \cdot \xi = \epsilon\xi^2 < 0. \tag{1.15}$$

Hence, in both cases $\exists u \in V_+ : u \cdot \xi < 0$ and the proof is complete.

□

## 1.2 Distribution Theory

### 1.2.1 The Space of Linear Forms (Dual Space)

**Theorem 1.2.1.** *Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$ and $\langle u, \rangle : \mathcal{V} \to \mathbb{F}$ (or $u : \mathcal{V} \to \mathbb{F}$) be a linear form from $\mathcal{V}$ to $\mathbb{F}$, then the set of all linear forms from $\mathcal{V}$ to $\mathbb{F}$, denoted by $Hom(\mathcal{V}, \mathbb{F})$, forms a vector space with addition and multiplication by scalars defined by*

- $\langle au, \phi \rangle = a\langle u, \phi \rangle$ *if* $a \in \mathbb{F}$ *and* $u \in Hom(\mathcal{V}, \mathbb{F})$ $\forall \phi \in \mathcal{V}$
- $\langle u + v, \phi \rangle = \langle u, \phi \rangle + \langle v, \phi \rangle$ *if* $u, v \in Hom(\mathcal{V}, \mathbb{F})$ $\forall \phi \in \mathcal{V}$

*Proof.* The proof that $Hom(\mathcal{V}, \mathbb{F})$ satisfies the 8 axioms for vector spaces is quite straightforward, even so, we shall prove two of them: the existence of identity and inverse elements. Let $\mathcal{O} : \mathcal{V} \to \mathbb{F}$ be the linear form with $Ker\ \mathcal{O} = \mathcal{V}$, that is, $\mathcal{O}(\phi) = \langle \mathcal{O}, \phi \rangle = 0 \quad \forall \phi \in \mathcal{V}$.

[7] For the definition of causal complement consult the definition A.0.1 in appendix A

[8] Small enough in the sense that $u$ is still a future-pointing time-like vector. Such an $\epsilon$ exists due to the fact that $V_+$ is an open convex subset of $\mathcal{M}$.

Then, $\forall u \in Hom(\mathcal{V}, \mathbb{F})$

$$\begin{aligned}\langle u + \mathcal{O}, \phi\rangle &= \langle u, \phi\rangle + \langle \mathcal{O}, \phi\rangle \\ &= \langle u, \phi\rangle + 0 \\ &= \langle u, \phi\rangle \quad \forall \phi \in \mathcal{V}. \end{aligned} \tag{1.16}$$

This proves that the linear form $\mathcal{O}$ is the identity element in $Hom(\mathcal{V}, \mathbb{F})$. To prove the existence of inverse elements, let $u \in Hom(\mathcal{V}, \mathbb{F})$. From the definition of scalar multiplication, we have $\langle -u, \phi\rangle = \langle -1 \cdot u, \phi\rangle = -\langle u, \phi\rangle \quad \forall \phi \in \mathcal{V}$. Hence,

$$\begin{aligned}\langle u + (-u), \phi\rangle &= \langle u, \phi\rangle + \langle -u, \phi\rangle \\ &= \langle u, \phi\rangle - \langle u, \phi\rangle \\ &= 0 \quad \forall \phi \in \mathcal{V}. \end{aligned} \tag{1.17}$$

$\square$

**Remark 1.2.2.** *A distribution is essentially a particular type of linear form defined on $C_c^\infty(\mathbb{R}^n)$ (also denoted by $\mathcal{D}(\mathbb{R}^n)$), which is the vector space of all complex-valued functions on $\mathbb{R}^n$ which possess continuous derivatives of all orders and vanish outside some bounded set. Moreover, a distribution is not actually the whole $Hom(C_c^\infty(\mathbb{R}^n), \mathbb{C})$, but a subspace (more precisely, isomorphic to a subspace) of it consisting of continuous linear forms. The term "continuous" presupposes the existance of an underlying topology. To define a distribution properly one should use the theory of locally convex topological spaces, however, to be parsimonious, we are not going to enter this beautiful and elegant realm of mathematics.*

Before defining a distribution and describing some of its fundamental properties we are going to succinctly characterize its domain, which is the space of test functions.

### 1.2.2 The Space of Test Functions

**Definition 1.2.3.** *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{C}$ be a continuous complex-valued function. The **support of** $f$, denoted symbolically by supp($f$) is the closure of the set of points in $\Omega$ where $f$ is non-zero, that is*

$$\begin{aligned} supp(f) :&= \overline{\{x \in \Omega : f(x) \neq 0\}} \\ &= \Omega \backslash \{y \in \Omega : \exists \text{ neighborhood } U \ni y : f|_U = 0\}. \end{aligned}$$

**Remark 1.2.4 (Properties).** *Let $f, g$ be functions on $\Omega \subseteq \mathbb{R}^n$, then*

1. *supp*$(f') \subseteq$ *supp*$(f)$;

2. *supp*$(fg) =$ *supp*$(f) \cap$ *supp*$(g)$;

3. *supp*$(f+g) \subseteq$ *supp*$(f) \cup$ *supp*$(g)$.

**Definition 1.2.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The space of complex-valued infinitely differentiable functions on $\Omega$ with compact support, denoted by $C_c^\infty(\Omega)$ (or $\mathcal{D}(\Omega)$), is called the* ***space of test functions on*** *$\Omega$.*

**Definition 1.2.6.** *A sequence of functions $(\varphi_j) \subset C_c^\infty(\Omega)$ is said to converge to a function $\varphi \in C_c^\infty(\Omega)$ as $j \to \infty$ if*

- *there exists a compact set $K \subset \Omega$ such that supp$(\varphi_j) \subseteq K$, $\forall j \in \mathbb{N}$, and*
- *the derivatives of any given order $m$ of the $\varphi_j$ converge uniformly as $j \to \infty$ to the corresponding derivative of $\varphi$*

### 1.2.3 Basic Definitions and Properties

**Definition 1.2.7.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A continuous linear functional $u : C_c^\infty(\Omega) \to \mathbb{C}$ is called a* ***distribution*** *in $\Omega$. That is, let $\chi_1, \chi_2 \in C_c^\infty(\Omega)$, $\lambda \in \mathbb{C}$ and $(\varphi_j)$ a sequence in $C_c^\infty(\Omega)$, then, if $u$ satisfies:*

- $u(\chi_1 + \lambda\chi_2) = u(\chi_1) + \lambda u(\chi_2)$;
- *if $\varphi_j \to 0$ in $C_c^\infty(\Omega)$, then $u(\varphi_j) \to u(0) = 0$,*

*then $u$ is a distribution. The space of distributions in $\Omega$ is denoted by $\mathcal{D}'(\Omega)$ (which is Schwartz' original notation [22]).*

**Example 1.2.8.** *Let $\chi_1, \chi_2, \varphi_j \in C_c^\infty(\mathbb{R})$. The classical* ***Dirac delta "function"****, defined by $\langle \delta, \varphi \rangle = \varphi(0)$, can be easily verified to be a distribution. Linearity follows from*

$$\begin{aligned} \langle \delta, \chi_1 + \lambda\chi_2 \rangle &= (\chi_1 + \lambda\chi_2)(0) \\ &= \chi_1(0) + \lambda\chi_2(0) \\ &= \langle \delta, \chi_1 \rangle + \lambda \langle \delta, \chi_2 \rangle, \end{aligned}$$

*and if $\varphi_j \to 0$ in $C_c^\infty(\mathbb{R})$, then*

$$\langle \delta, \varphi_j \rangle = \varphi_j(0) \to 0,$$

*which proves the continuity of $\delta$.*

**Example 1.2.9.** *Let $f \in L^1_{loc}(\Omega)$[9], where $\Omega$ is open in $\mathbb{R}^n$, then $u_f$ defined by*

$$u_f(\varphi) = \langle u_f, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \forall \varphi \in C_c^\infty(\Omega) \tag{1.18}$$

*is a distribution.*

*In fact, that is relatively easy to check. Let $\varphi, \varphi' \in C_c^\infty(\Omega)$ and $\lambda \in \mathbb{C}$, then linearity follows from*

$$\begin{aligned}\langle u_f, \varphi + \lambda\varphi' \rangle &= \int f(x)(\varphi(x) + \lambda\varphi'(x))dx \\ &= \int f(x)\varphi(x)dx + \lambda \int f(x)\varphi'(x)dx = \langle u_f, \varphi \rangle + \lambda \langle u_f, \varphi' \rangle,\end{aligned}$$

*and to prove the continuity of $u_f$, let $\{\varphi_j\} \subset C_c^\infty(\Omega)$ be a sequence, with $\varphi_j \to 0$ in $C_c^\infty(\Omega)$, then*

$$\begin{aligned}|\langle u_f, \varphi_j \rangle| &= \left| \int_{supp(\varphi_j)} f(x)\varphi_j(x)dx \right| \\ &\leq \sup_{x \in K} |\varphi_j(x)| \int_K |f(x)|\, dx \end{aligned} \tag{1.19}$$

*where $K$ is a compact set containing the supports of all $\varphi_j$, that is $\cup_{j\in\mathbb{N}} supp(\varphi_j) \subseteq K$. If $\varphi_j \to 0$, then the right-hand side of equation (1.19) approaches 0 and we are done.*

**Remark 1.2.10.** *Although in the previous example it was shown that a function $f \in C_c^\infty(\Omega)$ defines a distribution $u_f$, the distribution so defined cannot determine $f$ uniquely, since the right-hand side of equation (1.18) does not change if we replace $f$ by a function $g$ that is equal to $f$ almost everywhere (this means that the set $\{x \in \Omega : \ f(x) - g(x) \neq 0\}$ is a set of measure zero[10]). However, it follows that, if $f$ is locally integrable and the right-hand side of equation (1.18) vanishes $\forall \varphi \in C_c^\infty(\Omega)$, then $\langle u_f, \varphi \rangle = 0$ almost everywhere. In particular, the Dirac $\delta$-function treated in Example 1.2.8 is clearly not locally integrable since it is zero almost everywhere and $\forall f \in L^1_{loc}(\Omega)$, if the support of $f$ is a set of zero measure, then equation (1.18) is identically zero. By definition, the vector space $L^1_{loc}(\Omega)$ consists of all equivalence classes of locally integrable functions on $\Omega$.*

**Remark 1.2.11.** *From (1.18) we see that we can identify every locally integrable function with a distribution $u_f$ (not uniquely of course). Analagously, we can identify other spaces, such as $L^p(\Omega)$ [11] and $C^k(\Omega)$, with certain subspaces of $\mathcal{D}'(\Omega)$. It is in this very sense that we think of distributions as generalized functions.*

---

[9] $L^1_{loc}(\Omega)$ represents the space of functions which are integrable on compact subsets of $\Omega$.

[10] A set of measure zero in $\mathbb{R}^n$ is one that can, for every $\epsilon > 0$, be covered by a countable family of *n-cubes* with *n-volume* less than $\epsilon$.

[11] Informally speaking, an $L^p$ space may be defined as a space of functions for which the p-th power of the absolute value is Lebesgue integrable and the $C^k$ space consists of continuous differentiable functions of order $k$.

**Remark 1.2.12.** *Let $f$ be a continuously differentiable function on $\Omega \subseteq \mathbb{R}^n$, then*

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \forall \varphi \in C_c^\infty(\Omega)$$

*defines a distribution on $C_c^\infty(\Omega)$. Also, let $\partial/\partial x_i$ be the i-th derivative operator with $1 \leq i \leq n$, then we have*

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle &= \int \frac{\partial f}{\partial x_i}\varphi dx \\
&= \int \left[ \frac{\partial}{\partial x_i}(f\varphi) - f\frac{\partial \varphi}{\partial x_i} \right] dx \\
&= -\int f\frac{\partial \varphi}{\partial x_i}dx = -\left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad \forall \varphi \in C_c^\infty(\Omega)
\end{aligned} \tag{1.20}$$

*From the second to the third step in (1.20) we used the fact that $\varphi$ vanishes outside a bounded set. Since the right-hand side of equation (1.20) makes sense even for a non-differentiable function $f$, we may use it to define the derivative of $f$ as a distribution, that is*

$$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle \doteq -\left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad 1 \leq i \leq n, \ \forall \varphi \in C_c^\infty(\Omega). \tag{1.21}$$

*Essentially, by iterating (1.20) one obtains generalized derivatives of all orders.*

*Using a similar ideia to the definition of the derivative of a distribution, one can also define multiplication of a distribution by an infinitely differentiable function $g$ by setting*

$$\langle gu, \varphi \rangle = \langle u, g\varphi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.22}$$

*and more generally, combining equations (1.22) and (1.20) (in the generalized version), one can account for the action of any linear differential operator with smooth coefficients on $u$. That is, defining the operator*

$$\begin{aligned}
P(\partial) &:= \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n} g_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \frac{\partial^{|(\alpha_1,\alpha_2,\ldots,\alpha_n)|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} \\
&= \sum_{|\alpha|\geq 0} g_\alpha \partial^\alpha, \quad f_\alpha \in \ C^\infty(\mathbb{R}^n), \ \forall \alpha \in \mathbb{N}^n
\end{aligned} \tag{1.23}$$

*where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| \doteq \sum_{j=1}^n \alpha_j$, we have*

$$\begin{aligned}
\langle P(\partial)u, \varphi \rangle &= \left\langle \sum_{|\alpha|\geq 0} g_\alpha \partial^\alpha u, \varphi \right\rangle \\
&= \left\langle u, \sum_{|\alpha|\geq 0} (-1)^{|\alpha|} g_\alpha \partial^\alpha \varphi \right\rangle, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n)
\end{aligned} \tag{1.24}$$

### 1.2.4 Localization

In order to compare two functions we may apply them to some specific points in their domain. Likewise, to compare distributions we may look at their localizations, which are basicly their restriction to certain subsets of where they are defined. A distribution can be recovered from its localizations. In order to prove this, one has to indroduce the concept of partition of unity.

**Definition 1.2.13.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. A sequence of functions $\{\varphi_j\}_{j=1}^{\infty} \subset C_c^{\infty}(\Omega)$ is called a partition of unity of $\Omega$ if $\forall j \in \mathbb{N}$, we have*

1. *every point $x \in \Omega$ has a neighborhood that intersects only a finite number of the supports, supp $\varphi_j$;*
2. *$\sum_{j=1}^{\infty} \varphi_j(x) = 1$ for every $x \in \Omega$;*
3. *$0 \leq \varphi_j(x) \leq 1$ for every $x \in \Omega$.*

**Theorem 1.2.14.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and $K$ be a compact subset of $\Omega$. If $\Omega_i \subset \Omega$ for $i = 1, ..., n$ such that $\{\Omega_i\}_{i=1}^{n}$ is an open cover of $K$, then $\exists \varphi_i \in C_c^{\infty}(\Omega_i)$ for $i = 1, ..., n$ such that*

1. *$\sum_{i=1}^{n} \varphi_i(x) \leq 1$ on $\Omega$;*
2. *$\sum_{i=1}^{n} \varphi_i(x) = 1$ on a neighborhood of $K$;*
3. *$0 \leq \varphi_i(x) \leq 1$ on $\Omega_i$.*

**Definition 1.2.15.** *Two distributions $u_1, u_2 \in \mathcal{D}'(\Omega)$ are said to be **equal in an open set** $\mathbf{U} \subseteq \Omega$ if $\forall \varphi \in C_c^{\infty}(U)$, we have*

$$\langle u_1, \varphi \rangle = \langle u_2, \varphi \rangle \,. \tag{1.25}$$

**Theorem 1.2.16.** *Let $u_1, u_2 \in \mathcal{D}'(\Omega)$ such that every $x \in \Omega$, has a neighborhood $V_x$ where $u_1 = u_2$. Then, $u_1 = u_2$ in $\Omega$.*

**Definition 1.2.17.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathcal{D}'(\Omega)$. The **support of** $\mathbf{u}$, written as supp $u$, is the complement of the set*

$$\{x \in \mathbb{R}^n : u = 0 \text{ on a neighborhood of } x\}. \tag{1.26}$$

**Remark 1.2.18.** *Whenerver "$u = 0$ on a neighborhood of $x$" we may say that $u$ is **locally zero** at $x$. From the previous definition we may conclude that for every point $y$ sufficiently close to $x$, $u$ is locally zero at $y$. From which it follows that the set in (1.26) is a union of open sets, and is thus open, and since the support of $u$ is the complement of (1.26), it must be closed in $\Omega$.*

**Example 1.2.19.** *The support of the Dirac $\delta$-function is the orgin, that is supp $u = \{0\}$, and one has $\delta = 0$ on $\mathbb{R}^n \setminus \{0\}$. More generally, let a distribution $u$ be defined in terms of the $\delta$-function as*

$$u = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta, \tag{1.27}$$

*where $m \in \mathbb{N}$ and $c_\alpha \in \mathbb{R}$. Then, supp $u = \{0\}$. The converse also holds.*

**Remark 1.2.20.** *The subspace of $\mathcal{D}'(\Omega)$, with $\Omega \subset \mathbb{R}^n$ open, of compact distributions shall be denoted by $\mathcal{E}'(\Omega)$. Moreover, $\mathcal{E}'(\Omega)$ may be thought of as the space of distributions whose domain is the space of infinitely differentiable functions on $\Omega$, i.e. $C^\infty(\Omega)$.*

**Remark 1.2.21.** *In studying distributions an interesting question arises: Is it possible to "multiply" distributions? The general answer to this question is a resounding no! That is so due to the fact that, it is, usually, senseless to "evaluate" a distribution at some point. However, motivated, largely by questions in theoretical physics, we may ask under which circumstances it is possible to extend the notion of product of ordinary functions to product of distributions. To exemplify the impossibility of such a task in the broad sense, let us try to construct the product of the $\delta$-function with itself. In order to do that, consider the family of functions $\chi_\epsilon : \mathbb{R} \to \mathbb{R}$ with $\epsilon > 0$ given by*

$$\chi_\epsilon = \begin{cases} \frac{1}{\epsilon} & \text{if} \quad |x| \leq \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases} \quad , \tag{1.28}$$

*and let $f \in C_c^\infty(\mathbb{R})$. Then,*

$$\langle \chi_\epsilon, f \rangle = \int_{\mathbb{R}} \chi_\epsilon(x) f(x) dx = \frac{1}{\epsilon} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} f(x) dx \tag{1.29}$$

$$= \frac{1}{\epsilon}(\epsilon f(0) + O(\epsilon^3)) = f(0) + O(\epsilon^2) \tag{1.30}$$

*and therefore*

$$\lim_{\epsilon \to 0} \langle \chi_\epsilon, f \rangle = f(0) = \langle \delta, f \rangle. \tag{1.31}$$

*We may conclude that $\lim_{\epsilon \to 0} \chi_\epsilon = \delta$ in the distributional sense. However, the square o $\chi_\epsilon$ does not converge to a distribution, since*

$$\langle \chi_\epsilon^2, f \rangle = \int_{\mathbb{R}} \chi_\epsilon^2(x) f(x) dx = \frac{1}{\epsilon^2} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} f(x) dx \tag{1.32}$$

$$= \frac{1}{\epsilon^2}(\epsilon f(0) + O(\epsilon^3)) = \frac{f(0)}{\epsilon} + O(\epsilon), \tag{1.33}$$

*which clearly diverges as $\epsilon \to 0$ if $f(0) \neq 0$.*

*In 1.2.17 the support of a distribution is defined. There is a similar notion, called the **singular support** of a distribution $u$, denoted symbolically by sing supp $u$, which can be defined in the following way: Let, $u \in \mathcal{D}'(\mathbb{R}^n)$, then $x \notin$ sing supp $u$ if and only if* *there exists a neighborhood $V$ of $x$, such that the restriction of $u$ to $V$ is a smooth function, that is, $\exists f \in C^{\infty}(V)$ such that $\forall \varphi \in C_c^{\infty}(\mathbb{R}^n)$, supp $u \subset V$. With this definition one can make sense of the product of two distributions if their singular supports are disjoint [23]. There is a yet more general way of coherently defining the product of two distributions, involving the notion called **the wavefront set**[12].*

[12] We are not going to use this notion throughout the thesis, but for a good reference in this matter see [23]

# 2 Geometric Time-ordering

In a given Lorentz frame $\{e_{(\mu)}, \mu = 0, \ldots, 3\}$, the time-coordinate of an event $x$ is just $x \cdot e_{(0)}$, and an event $x$ occurs "later" than an event $y$ in this frame if $(x - y) \cdot e_{(0)} > 0$. We therefore say that $x$ is later than $y$ if there is some time-like future-pointing vector $u$ such that $(x - y) \cdot u > 0$. By Theorem 1.1.16, this is equivalent to the condition that $x$ be outside the closed backward light cone of $y$. (For general geometric definitions and conventions that will be used throughout the rest of this work, see Appendix A.) We take this as a definition:

**Definition 2.0.1** (Geometric time-ordering)**.** *For $x, y \in \mathbb{R}^4$ we say that $x$ is later than $y$, in symbols $x \succcurlyeq y$, if $x$ is not contained in the past light-cone of $y$:*

$$x \succcurlyeq y \;\; :\Leftrightarrow \;\; x \notin \overline{V_-(y)}. \tag{2.1}$$

*For subsets $R, S \subset \mathbb{R}^4$, we say that $R$ is later than $S$, symbolically $R \succcurlyeq S$, if all points in $R$ are later than all points in $S$. If either $R \succcurlyeq S$ or $S \succcurlyeq R$, we say $R$ and $S$ are comparable; otherwise, we say $R$ and $S$ are incomparable and write $R \not\gtrless S$.*

(For a point $x \in \mathbb{R}^4$ and a region $R \subset \mathbb{R}^4$ we write sloppily $x \succcurlyeq R$ instead of $\{x\} \succcurlyeq R$.)

There are two warnings in order: Physically, the time-ordering relation $x \succcurlyeq y$ must be distinguished from the causality relation $x \in \overline{V_+(y)}$, which means that $y$ can influence the event $x$ either by way of the propagation of some material phenomenon or some electromagnetic effect. Mathematically, "$\succcurlyeq$" is not an order relation: It is not transitive (see Fig. 1 for a counter-example), and it is not linear, namely not every pair of regions is comparable.

## 2.1 Generalities

We establish some properties of the time-ordering relation which are relevant for the proof of Propositions 2.2.1 and 2.2.4. First, note that for two regions $R, S$ in Minkowski space we have

$$R \succcurlyeq S \quad \Leftrightarrow \quad R \cap \overline{V_-(S)} = \emptyset. \tag{2.2}$$

**Lemma 2.1.1.** *Let $R, S \subset \mathbb{R}^4$.*
*i) There holds both $R \succcurlyeq S$ and $S \succcurlyeq R$, if, and only if, $R$ and $S$ are space-like separated.*
*ii) $R$ and $S$ are incomparable if, and only if, both $R \cap \overline{V_-(S)} \neq \emptyset$ and $R \cap \overline{V_+(S)} \neq \emptyset$.*

*Proof.* Note first that $R \cap \overline{V_-(S)} = \emptyset$ is equivalent to $S \cap \overline{V_+(R)} = \emptyset$. Thus, $(S \succcurlyeq R) \wedge (R \succcurlyeq S)$ is equivalent, by Eq. (2.2), to $S \cap \overline{V_-(R)} = \emptyset \wedge S \cap \overline{V_+(R)} = \emptyset$. But this is $S \cap \left(\overline{V_+(R)} \cup \overline{V_-(R)}\right) = \emptyset$, which means just that $S$ is space-like separated from $R$. This proves $i$). $ii$) is a direct consequence of Eq. (2.2). □

**Lemma 2.1.2.** *Let $\Sigma$ be a space-like hyperplane of the form $\Sigma = a + u^\perp$, where $u$ is a future-pointing time-like vector and $a \in \Sigma$. Then, for all $x \in \mathbb{R}^4$ there holds $x \succcurlyeq \Sigma$ if, and only if, $(x - a) \cdot u > 0$, that is $x$ is "above" $\Sigma$.*

*Proof.* Firstly, note that the condition $x \succcurlyeq \Sigma$ means, by definition, that $\forall y \in \Sigma$, $x - y \notin \overline{V_-}$. Moreover, $\forall y \in \Sigma$ there holds $u \cdot (a - y) = 0$, and consequently

$$u \cdot (x - y) = u \cdot (x - a).$$

Now suppose $u \cdot (x - a) > 0$, and let $y \in \Sigma$, then $u \cdot (x - y) \equiv u \cdot (x - a) > 0$, which implies $x - y \notin \overline{V_-}$. This shows that $x \succcurlyeq \Sigma$. We prove inverse direction contrapositively. Suppose, that $u \cdot (x - a) \leq 0$. If $u \cdot (x - a) = 0$, then $x \in \Sigma$ and thus $\neg(x \succcurlyeq \Sigma)$. On the other hand, $u \cdot (x - a) < 0$ implies as above that $x \preccurlyeq \Sigma$. But then $x$ cannot be later than $\Sigma$ by Lemma 2.1.1, since the causal complement of $\Sigma$ is empty. In both cases, we have $\neg(x \succcurlyeq \Sigma)$. Summarizing, we have shown that $x \succcurlyeq \Sigma$ is equivalent to $(x - a) \cdot u > 0$. □

The (motivating) characterization of the relation $x \succcurlyeq y$, namely the condition that $x^0 > y^0$ in some reference frame, can now be written as the condition that there exists a space-like hyperplane $\Sigma$ that is separating in the sense that $x \succcurlyeq \Sigma \succcurlyeq y$. As we have seen, this is equivalent to our Definition 2.0.1. The same holds for finite string segments.

However, for infinitely extended strings, the existence of a space-like separating hyperplane is a sufficient but not necessary condition for time-ordering as defined in Def. 2.0.1. (A sufficient and necessary condition would be the existence of a space-like *or light-like* separating hyperplane. But we don't need this statement and therefore refrain from proving it.)

**Lemma 2.1.3.** *Let $R_1 \subseteq S_1, R_2 \subseteq S_2$ be two subsets of the strings $S_1$ and $S_2$ (which comprises points, finite string segments and the entire string, exhausting all possiblities). If there is a space-like hyperplane $\Sigma$ such that $R_1 \succcurlyeq \Sigma \succcurlyeq R_2$, then $R_1 \succcurlyeq R_2$.*

*Proof.* Let $\Sigma := a + u^\perp$ satisfy the hypothesis, and let $z_1 \in R_1$ and $z_2 \in R_2$. Then by Lemma 2.1.2 there holds $(z_1 - a) \cdot u > 0$ and $(a - z_2) \cdot u > 0$. Adding these two inequalities yields $(z_1 - z_2) \cdot u > 0$, and since $u \in V_+$ we must have $z_1 - z_2 \notin \overline{V_-}$ by Lemma 1.1.16 (with reversed signs), that is $z_1 \succcurlyeq z_2$. This completes the proof. □

### 2.1.1 Comparability

For points, the time ordering relation is linear in the sense that any pair of distinct events $x \neq y \in \mathbb{R}^4$ is comparable, *i.e.*, there holds either $x \succcurlyeq y$ or $y \succcurlyeq x$. The first problem we encounter in the definition of time-ordered products is that this is not so for disjoint strings: It may happen that one string enters into the past *and* into the future of the other one, and in this case (and only in this case) the two strings are not comparable by Lemma 2.1.1, (*ii*). On the other hand, a point and a string are always comparable given that they are disjoint:

**Lemma 2.1.4.** *Let $S$ be a string and $x \in \mathbb{R}^4 \setminus S$ a point disjoint from $S$. Then either $S \succcurlyeq \{x\}$ or $\{x\} \succcurlyeq S$.*

*Proof.* Suppose that neither $S \succcurlyeq x$ nor $x \succcurlyeq S$ holds. Then, by Eq. (2.2), both $S \cap \overline{V_-(\{x\})}$ and $\{x\} \cap \overline{V_-(S)}$ are not empty. However, since $S \cap \overline{V_-(\{x\})} \neq \emptyset \Leftrightarrow \{x\} \cap \overline{V_+(S)} \neq \emptyset$, we must have $x \in \overline{V_-(S)} \cap \overline{V_+(S)} = S$, and the proof is complete by contraposition. □

### 2.1.2 Transitivity

Time-ordering of events is not transitive. But it has a similar property, which we might call "weak transitivity": If $y_1 \succcurlyeq y_2$ and $x \not\succcurlyeq y_2$, then $y_1 \succcurlyeq x$. This fact is the basis for the proof that Bogoliubov's S-matrix satisfies the functional equation [13], which in turn implies locality of the interacting fields in the Epstein-Glaser construction [13]. Again, this does not hold for strings – This is why a string-localized interaction leads in general to completely non-local interacting fields. An example is illustrated in Fig. 1, which shows three strings satisfying $S_1 \succcurlyeq S_2$ and $S_3 \not\succcurlyeq S_2$, however $S_1$ is not later than $S_3$.

On the other hand, weak transitivity does hold for two strings with respect to an event:

**Lemma 2.1.5.** *Let two strings $S_1, S_2$ and an event $x \in \mathbb{R}^4 \setminus (S_1 \cup S_2)$ be such that*

$$S_1 \succcurlyeq S_2 \quad \text{and} \quad x \not\succcurlyeq S_2.$$

*Then $S_1 \succcurlyeq \{x\}$.*

*Proof.* By Eq. (2.2), the premise $S_1 \succcurlyeq S_2$ may be written as $S_1 \cap \overline{V_-(S_2)} = \emptyset$. Also, $x \not\succcurlyeq S_2$ means $x \in \overline{V_-(S_2)}$ and consequently $\overline{V_-(x)} \subset \overline{V_-(S_2)}$. Thus, we must have $S_1 \cap \overline{V_-(x)} = \emptyset$, and again by Eq. (2.2), $S_1 \succcurlyeq \{x\}$. □

### 2.1.3 Latest Member

**Definition 2.1.6.** *Given $n$ subsets $R_1, \ldots, R_n$ of Minkowski space, we say that $R_1$ is a* latest member *of the set $\{R_1, \ldots, R_n\}$ if $R_1 \succcurlyeq R_i$ for all $i = 2, \ldots, n$.*

A basic fact is that $n$ distinct events in Minkowski space always have a latest member, and the same holds for sufficiently small neighborhoods of them. Again, this is not so for strings (see the counter-example in Fig. 1).

Lemma 2.1.5 implies that two comparable strings and one event, disjoint from the strings, always have a latest member. (Namely, if $x$ is later than both $S_1$ and $S_2$ then it is of course $x$, and otherwise it is the later one of the strings.) For our chopping of $n > 2$ strings, we need more:

**Lemma 2.1.7.** *Let $S_1, \ldots, S_r$ be strings which have a latest member, and let $y_1, \ldots, y_k$ be pairwise distinct points in Minkowski space satisfying $y_i \notin S_j$ for all $1 \leq i \leq k$ and $1 \leq j \leq r$. Then, the set $\{S_1, \ldots, S_r, \{y_1\}, \ldots, \{y_k\}\}$ also has a latest member.*

*Proof.* We assume that $S_1$ is a latest member of the strings.

First case: One of the points, say $y_1$, is later than all the strings. Let $y_l$ be a latest member of $\overline{V_+(y_1)} \cap \{y_1, \ldots, y_r\}$, the subset of $y's$ which lie to the future of $y_1$. Then $y_l$ is a latest member of *all* the $y$'s, and also later than all the strings[1], that is, $y_l$ is a latest member of $\{S_1, \ldots, S_r, \{y_1\}, \ldots, \{y_k\}\}$.

Second case: None of the $y$'s is later than all the strings. Then for every $i \in \{1, \ldots, k\}$ there is a $j(i) \in \{1, \ldots, r\}$ such that $y_i \not\succcurlyeq S_{j(i)}$. If $j(i) = 1$ (the label of the latest member of the strings $\{S_1, \ldots, S_r\}$), then $y_i \not\succcurlyeq S_1$, and Lemma 2.1.4 implies that $S_1 \succcurlyeq y_i$. If $j(i) \neq 1$, then $S_1 \succcurlyeq S_{j(i)}$ and Lemma 2.1.5 implies that $S_1 \succcurlyeq y_i$. Summarizing, in the second case for all $i$ there holds $S_1 \succcurlyeq y_i$. Then $S_1$ is a latest member of $\{S_1, \ldots, S_r, \{y_1\}, \ldots, \{y_k\}\}$. □

## 2.2 String Chopping

As mentioned in the introduction, we wish to show that one can chop $n$ strings into small enough pieces which are mutually comparable. We first give a constructive prove for $n = 2$, where it suffices to cut one of the strings once.

By *cutting* of a string $S = S_{x,e}$ is meant the selection of one point $x + se$ for some $s > 0$, whereby the string becomes a union of the finite segment $x + [0, s]\, e$ and the residual string $x + [s, \infty)\, e$. The two pieces do not overlap, since they have only the cut point in common. We write this nonoverlapping union as $S = S^1 \cup S^2$, but it suits us not to specify which piece is the finite segment and which is the tail.

**Proposition 2.2.1.** *Let $S, S'$ be two disjoint strings. Then there is a chopping of $S$ into two pieces (one segment and one string) $S = S^1 \cup S^2$, such that both pairs $(S^1, S')$ and $(S^2, S')$ are comparable.*

[1] Here we use the obvious fact that $\left(y_1 \succcurlyeq S \wedge y_2 \in \overline{V_+(y_1)}\right) \Rightarrow y_2 \succcurlyeq S$.

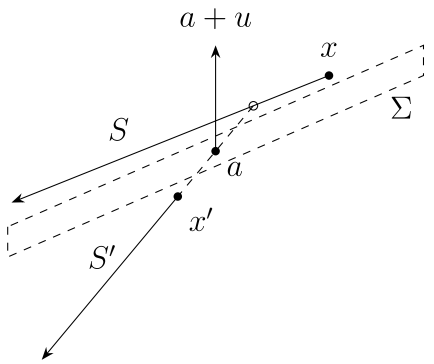


**Figure 2** – The strings $S$ and $S'$, with $\widetilde{S'}$ meeting $S$

*Proof.* Let $S = S_{x,e}$ and $S' = S_{x',e'}$, and denote by $\widetilde{S'} \doteq S_{x',e'} \cup S_{x',-e'}$ the full straight line through $x'$ with direction $e'$.

We first consider the case when $S$ meets $\widetilde{S'}$ (but is disjoint from $S'$). Then there are positive reals $t, t'$ such that

$$x + te = x' - t'e'. \tag{2.3}$$

Suppose the span of $e, e'$ is space-like. Then $S$ and $S'$ are disjoint sets contained in the space-like hyperplane $x + \mathrm{span}\{e, e'\}$. This implies that $S, S'$ are space-like separated, and thus comparable (see Lemma 2.1.1, item $i$)). No chopping is needed.

Suppose now that the span of $e, e'$ is time- or light-like. Then Lemma A.0.4 implies that one of the vectors $e \pm e'$ is time- or light-like. Suppose first that $e - e'$ is a time- or light-like vector, and assume that it is future oriented, $e \in \overline{V_+(e')}$.Let $u$ be any timelike future-oriented vector orthogonal to $e$, and (see Fig. 2) let

$$\Sigma \doteq a + u^\perp, \quad a \doteq x' - \frac{t'}{2}\, e'.$$

Now $e' \cdot u$ is strictly negative since $e \cdot u = 0$ and $e' \in \overline{V_-(e)}$, and we therefore get

$$\begin{aligned}(x + se - a) \cdot u &\equiv -\frac{t'}{2} e' \cdot u \qquad > 0,\\ (x' + s'e' - a) \cdot u &\equiv (s' + \frac{t'}{2}) e' \cdot u < 0.\end{aligned}$$

(In the first line we have used Eq. (2.3).) The two inequalities say that $S \succcurlyeq \Sigma$ and that $S' \preccurlyeq \Sigma$, respectively. By Lemma 2.1.3, this shows that $S \succcurlyeq S'$. If $e - e'$ is past oriented, $e \in \overline{V_-(e')}$, then the same argument shows that $S' \succcurlyeq S$. In the case when $e + e'$ (instead of $e - e'$) is a time- or light-like vector, the same argument goes through with $e'$ replaced by $-e'$. This completes the proof in the case when $S$ meets $\widetilde{S'}$.

For the rest of the proof we consider the case when $S$ is disjoint from $\widetilde{S'}$. First, assume that $S \cap (\tilde{S}')^c = \emptyset$, that is, $S$ is contained in the closure of $V_-(\widetilde{S'}) \cup V_+(\widetilde{S'})$. If $S$ had non-trivial intersection with both $\overline{V_-(\widetilde{S'})}$ and $\overline{V_+(\widetilde{S'})}$, it would have to pass through

$\widetilde{S'}$, which was excluded. Thus, $S$ is contained entirely in the closure of either $V_-(\widetilde{S'})$ or $V_+(\widetilde{S'})$, and in this case $S$ and $S'$ are comparable by Lemma 2.1.1, *ii*). No chopping is needed.

Now suppose that $S \cap (\widetilde{S'})^c \neq \emptyset$. If $e = \pm e'$ (i.e., the strings $\widetilde{S}$ and $\widetilde{S'}$ are parallel), then $S$ is completely contained in the causal complement of $\widetilde{S'}$, and thus $S$ is both later and earlier than $S'$, and no chopping is needed.

Consider finally the case $e \neq \pm e'$. The claim is that there exists a chopping $S = S_+ \cup S_-$, such that $S_+ \succcurlyeq S' \succcurlyeq S_-$. Using Lemma 2.1.3, it is sufficient to establish the existence of two space-like hyperplanes $\Sigma_1$, $\Sigma_2$, such that $S_+ \succcurlyeq \Sigma_1 \succcurlyeq S'$ and $S' \succcurlyeq \Sigma_2 \succcurlyeq S_-$.

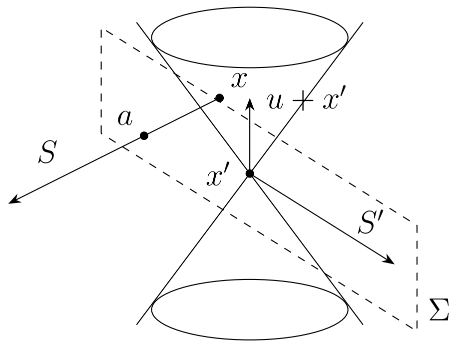


**Figure 3** – Selection of a cut point on the string $S$

Take an event $a \in S \cap (\widetilde{S'})^c$ with $a \neq x$; this is the place where we cut $S$ (see Fig. 3). The vector $a - x'$ is space-like and space-like separated from $e'$, hence the 2-plane $E \doteq \operatorname{span}\{a - x', e'\}$ is space-like. Choose a time-like future-directed vector $u$ in the orthogonal complement $E^\perp$, which is *not* orthogonal to $e$. (The possibility $u \cdot e \neq 0$ is allowed since $e \neq \pm e'$.) Note that $\Sigma \doteq u^\perp$ contains the string $S'$ and cuts the string $S$ through the point $a$. Our hyperplanes $\Sigma_1$, $\Sigma_2$ will be small transformations of $\Sigma$. First, we shift $\Sigma$ by a small amount so that it does not contain the point $a$ any more: Let $P_{e'}^\perp$ be the projector onto $(e')^\perp$, and let

$$u_\pm := u \pm \varepsilon P_{e'}^\perp (a - x'),$$

where $\varepsilon$ is small enough that

$$\operatorname{sgn}(u_\pm \cdot e) = \operatorname{sgn}(u \cdot e) =: \sigma. \tag{2.4}$$

Let now $\Sigma_\pm \doteq x' + (u_\pm)^\perp$, and define the chopping $S = S_+ \cup S_-$, where

$$S_\pm \doteq \Big(a \pm \sigma \mathbb{R}^+ e\Big) \cap S.$$

(If $\sigma > 0$, then $S_+$ is the "infinite tail" of the chopping while $S_-$ is a finite segment, and if $\sigma < 0$ the roles are interchanged.) Both $\Sigma_\pm$ still contain $S'$, and are in addition also comparable with $S_\pm$:

Claim.

$$S_+ \succcurlyeq \Sigma_- \text{ and } S_- \preccurlyeq \Sigma_+\,.$$

*Proof of claim.* Let $\xi \doteq a - x'$. By Lemma 2.1.2, the first relation is equivalent to

$$(\xi + s\sigma e)\cdot u_- \equiv -\varepsilon\xi\cdot P^\perp_{e'}(\xi) + s|e\cdot u_-| \;>\; 0 \text{ for all } s \geq 0. \tag{2.5}$$

(We have used $(a-x')\cdot u = 0$ and Eq (2.4).) Note that the projector $P^\perp_{e'}$ is given by

$$P^\perp_{e'}(\xi) = \xi - \frac{e'\cdot\xi}{e'\cdot e'}\,e' = \xi + (e'\cdot\xi)e',$$

hence $\xi\cdot P^\perp_{e'}(\xi) = \xi\cdot\xi + (e'\cdot\xi)^2$. Now the condition that $a \in (\widetilde{S'})^c$ means that $\xi - te'$ is space-like for all $t\in\mathbb{R}$. Thus, the quadratic form $-t^2 - 2t(e'\cdot\xi) + \xi\cdot\xi \equiv -(t+e'\cdot\xi)^2 + \xi\cdot\xi + (e'\cdot\xi)^2$ is strictly negative, which implies that

$$(\xi\cdot\xi) + (e'\cdot\xi)^2 < 0; \text{ and thus } \xi\cdot P^\perp_{e'}(\xi) < 0. \tag{2.6}$$

This proves the inequality (2.5), and thus the first relation of the lemma. The second one is shown analogously: It means that

$$(\xi - s\sigma e)\cdot u_+ \equiv \varepsilon\xi\cdot P^\perp_{e'}(\xi) - s|e\cdot u_-| \;<\; 0 \text{ for all } s \geq 0, \tag{2.7}$$

which holds true by Eq. (2.6). □

We now shift the hyperplanes $\Sigma_\pm$ a little bit, so that they still satisfy the relations of the above lemma, and are in addition also comparable with $S'$. To this end, notice that the left hand side of the inequality (2.5) has the positive lower bound $\delta := -\varepsilon\xi\cdot P^\perp_{e'}\xi$. Thus, we can shift $\Sigma_-$ away from $S'$ by using instead

$$\Sigma'_- \doteq \Sigma_- + \alpha\, u_-, \quad \alpha \doteq \frac{\delta}{2u_-\cdot u_-}\,,$$

and the relation $S_+ \succcurlyeq \Sigma'_-$ still holds. On the other hand, for all $s>0$ there holds

$$(se' - \alpha u_-)\cdot u_- \equiv -\frac{1}{2}\delta < 0,$$

and therefore $S' \preccurlyeq \Sigma'_-$. We have now achieved $S_+ \succcurlyeq \Sigma'_- \succcurlyeq S'$, as required.

Similarly, one verifies that $\Sigma'_+ \doteq \Sigma_+ - \frac{\delta}{2u_+\cdot u_+}\,u_+$ satisfies $S_+ \preccurlyeq \Sigma'_+ \preccurlyeq S'$. This completes the proof. □

We now consider the case of $n>2$ strings. The *large string diagonal* is defined by

$$\Delta_n \doteq \Big\{(x_1,e_1,\ldots,x_n,e_n) \;\Big|\; S_{x_i,e_i}\cap S_{x_j,e_j} \neq \emptyset \;\text{ for some } i\neq j\Big\}. \tag{2.8}$$

We are going to show that $n$ strings outside $\Delta_n$ can be chopped up into finitely many pieces which are mutually comparable (Prop. 2.2.4). Here we shall need to cut the strings into more than two pieces. By a *chopping* of a string $S \doteq x + \mathbb{R}_0^+ e$ we mean a decomposition

$$S = S^{\text{fin}} \cup S^{\infty}, \qquad S^{\text{fin}} = \bigcup_{\alpha=1}^{N} S^{\alpha}, \tag{2.9}$$

determined by $N$ numbers $0 = s_0 < s_1 < \cdots < s_N$, where $S^{\alpha}$ is the finite segment

$$S^{\alpha} \doteq x + [s_{\alpha-1}, s_{\alpha}]e \tag{2.10}$$

and $S^{\infty}$ is the infinite tail of the string

$$S^{\infty} \doteq x + [s_N, \infty)e. \tag{2.11}$$

Before stating and proving Prop. 2.2.4, we need some lemmas. We start with considerations about the infinite tails of the strings $S_{x_i,e_i}$. If you look at $n$ strings from sufficiently far away, they seem to have their “heads” $x_i$ quite close to the origin (wherever you choose the origin). Hence, if you cut them far away from their heads, their infinite tails extend almost radially to infinity and thus correspond to points on the hyperboloid $H$ of space-like directions. Consequently, these tails can be linearly ordered, just like points in $H$ can. We realize this idea by showing first that every string $S_{x,e}$ eventually ends up in a space-like cone centered around the string $S_{0,e}$ with arbitrarily small opening angle. In detail, let $D$ be a neighborhood of $e$ in $H$, and let $C_D$ be the space-like cone

$$C_D \doteq \mathbb{R}^+ D = \{se' \mid s \in \mathbb{R}^+, \, e' \in D\} \tag{2.12}$$

centered at the origin.

**Lemma 2.2.2.** *For every string $S_{x,e}$ and every neighborhood $D$ of $e$ in $H$ there is an $s > 0$ such the infinite tail $x + [s, \infty)e$ is contained in the space-like cone $C_D$.*

*Proof.* Note that $y \in C_D$ if, and only if, $|y \cdot y|^{-1/2} y$ is in $D$. Thus, a point $x + te$ on the string is in $C_D$ iff the point $|(x + te) \cdot (x + te)|^{-1/2}(x + te)$ is in the neighborhood $D$. But this point can be written

$$\frac{t}{|x \cdot x + 2tx \cdot e - t^2|^{\frac{1}{2}}} \, (\frac{x}{t} + e),$$

which obviously converges to $e$ if $t \to \infty$. Thus, the curve $|(x + te)^2|^{-1/2}(x + te)$ eventually ends up in $D$, that is, $x + te$ eventually ends up in $C_D$. But this is just the claim. $\square$

This Lemma is relevant for time ordering due to the following fact.

**Lemma 2.2.3.** *Take two strings $S_1$, $S_2$ which are contained in space-like cones of the form (2.12), $S_i \subset C_{D_i}$, where $D_1$ and $D_2$ are double cones in the manifold $H$ of space-like directions. Suppose $D_1 \succcurlyeq D_2$, where the time-ordering on $H$ is defined in the same way as the time-ordering in Minkowski space (see Eq. (2.1)). Then $S_1 \succcurlyeq S_2$.*

*Proof.* Just as in Minkowski space (see Def. A.0.2), each double cone $D_i$ is characterized by its past and future tip, $e_i^+ \in V_+(e_i^-)$:

$$D_i = D_i(e_i^-, e_i^+) \doteq V_+(e_i^-) \cap V_-(e_i^+).$$

The hypothesis that $D_1 \succcurlyeq D_2$ obviously implies (in fact, is equivalent to) $e_1^- \succcurlyeq e_2^+$. To proceed, we first need an intermediate result: Not only that there exists a space-like hyperplane $\Sigma$ such that $e_1^- \succcurlyeq \Sigma \succcurlyeq e_2^+$, but that there is also one passing through the origin that does so.

Claim.

Let $e, e' \in H$ with $e \succcurlyeq e'$. Then there exists $u \in V_+$ such that $u \cdot e > 0$ and $u \cdot e' < 0$.

*Proof of claim.* The following four cases can occur:

1. The span of $e, e'$ is space-like. Then, by Lemma A.0.4, $e \cdot e' \in (-1, 1)$. Let $u \in \operatorname{span}\{e, e'\}^\perp$ be a future-pointing time-like vector and define $u_\varepsilon := u - \varepsilon(e - e')$ with $\varepsilon > 0$ small enough such that $u_\varepsilon$ is still in $V_+$. Then

$$u_\varepsilon \cdot e = \varepsilon(1 + e \cdot e') > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = -\varepsilon(1 + e \cdot e') < 0$$

2. The span of $e, e'$ is time-like. According to Lemma A.0.4, the following two cases can occur:

   a) The vector $e - e'$ is time-like. Then, since $e \succcurlyeq e'$ is assumed, it is future-pointing. Moreover, we must have $e \cdot e' < -1$. Thus, the vector $u \doteq e - e'$ does the job: $u \cdot e = -1 - e \cdot e' > 0$ and $u \cdot e' = e \cdot e' + 1 < 0$.

   b) The vector $e - e'$ is space-like and $e + e'$ is time-like. Then we must have $e \cdot e' > 1$. If $e + e'$ is future pointing, then

$$u \doteq P_{e'}^\perp(e) = e + (e' \cdot e)e'$$

   is time-like, since $u \cdot u = -1 + (e \cdot e')^2 > 0$. It is also future-pointing. (This can be seen as follows. Choose $v \in V_+$ with $v \cdot e = 0$. Then $v \cdot e' \equiv v \cdot (e + e') > 0$ since $e + e'$ is future-pointing.) Now put $u_\varepsilon \doteq u + \varepsilon e'$ for sufficiently small $\varepsilon > 0$. Then we have

$$u_\varepsilon \cdot e = -1 + (e' \cdot e)^2 + \varepsilon e \cdot e' > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = -\varepsilon < 0.$$

   If $e + e'$ is past pointing, then

$$u_\varepsilon \doteq -P_e^\perp(e') - \varepsilon e = -\big(e' + (e \cdot e')e\big) - \varepsilon e$$

   has the following properties, as the reader will readily verify: It is time-like and future-pointing, and satisfies

$$u_\varepsilon \cdot e = \varepsilon > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = 1 - (e' \cdot e)^2 - \varepsilon e \cdot e' < 0.$$

3. The span of $e, e'$ is light-like. From Lemma A.0.4, there are two possibilities:

   a) The vector $e - e'$ is light-like. Then it is future pointing by hypothesis, and moreover we must have $e \cdot e' = -1$. Choose $u \in (e')^\perp \cap V_+$ and let $u_\varepsilon \doteq u + \varepsilon e$ with sufficiently small $\varepsilon$. Then $u_\varepsilon$ is a future-pointing time-like vector satisfying

   $$u_\varepsilon \cdot e = u \cdot e - \varepsilon > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = -\varepsilon < 0.$$

   (We used that $u \cdot e \equiv u \cdot (e + e')$ is positive since $e + e'$ is assumed future-pointing; and we chose $\varepsilon$ small enough.)

   b) The vector $e + e'$ is light-like. Then we must have $e \cdot e' = 1$. If the vector $e + e'$ is future-pointing, the same $u_\varepsilon$ as in the above item does the job. If it is past pointing, pick $u \in e^\perp \cap V_+$ and let $u_\varepsilon \doteq u - \varepsilon e$ with sufficiently small $\varepsilon$. Then $u_\varepsilon$ is a future-pointing time-like vector satisfying

   $$u_\varepsilon \cdot e = \varepsilon > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = u \cdot e' + \varepsilon < 0.$$

   (We used that $u \cdot e' \equiv u \cdot (e + e')$ is negative since $e + e'$ is assumed past pointing; and we chose $\varepsilon$ small enough.)

4. In the last possible case, $e = -e'$, let $u \in e^\perp \cap V_+$ and define $u_\varepsilon \doteq u - \varepsilon e$. Then

   $$u_\varepsilon \cdot e = \varepsilon > 0 \quad \text{and} \quad u_\varepsilon \cdot e' = -\varepsilon < 0.$$

This proves the claim. □

We have thus shown that there exists a future-pointing time-like vector $u$ that satisfies $u \cdot e_1^- > 0 > u \cdot e_2^+$. It follows that $\forall e_1 \in D_1$ and $\forall e_2 \in D_2$ we have $u \cdot e_1 > 0 > u \cdot e_2$. This implies of course that $u \cdot re_1 > 0 > u \cdot se_2$ for $r, s \in \mathbb{R}^+$, and since all $z_i \in C_{D_i}$ are of the form $z_i = re_i$ with $r \in \mathbb{R}^+$ and $e_i \in D_i$, we have $C_{D_1} \succcurlyeq u^\perp \succcurlyeq C_{D_2}$ and consequently, by Lemma 2.1.3, $S_1 \succcurlyeq S_2$. □

We are now prepared for our main geometrical result:

**Proposition 2.2.4.** *Let $(\underline{x}, \underline{e})$ be outside the large string diagonal $\Delta_n$. Then there exists a chopping*[2]

$$S_{x_i, e_i} = \bigcup_{\alpha=1}^{N_i+1} S_i^\alpha,$$

*such that every selection $\{S_1^{\alpha_1}, \ldots, S_n^{\alpha_n}\}$ has a latest member, that is, for every $n$-tuple $(\alpha_1, \ldots, \alpha_n)$ there exists $i \in \{1, \ldots, n\}$ such that for every $j \in \{1, \ldots, n\} \setminus \{i\}$ there holds $S_i^{\alpha_i} \succcurlyeq S_j^{\alpha_j}$.*

[2] Here $S_i^{N_i+1}$ is the infinite tail of the string $S_i$, before denoted as $S_i^\infty$.

*Proof.* We first consider the infinite tails. Note that some of the $e$'s may coincide, so the *set* of $e$'s may contain less than $n$ (different) points. These have a latest member, and the same holds for sufficiently small double cones $D_i \ni e_i$ (understanding that $D_i = D_j$ if $e_i = e_j$). Let us denote the index of the latest member of $\{D_1, D_2, \ldots\}$ by $i_0$. Let further $s_i \in \mathbb{R}_0^+$ be such that the infinite tail $S_i^\infty \doteq x_i + [s_i, \infty)e_i$ of $S_i$ is contained in $C_{D_i}$ (see Lemma 2.2.2). By Lemma 2.2.3, the infinite tail with number $i_0$ is later than all the other ones. If $e_{i_0}$ coincides with $\{e_{i_1}, \ldots, e_{i_k}\}$, then the corresponding infinite tails are parallel and disjoint, and therefore have a latest member. (Since the problem can be reduced to distinct points in $e_{i_0}^\perp$.) This is the latest member of all infinite tails.

We now construct a chopping of the compact segments $S_i^{\rm fin} \doteq x_i + [0, s_i]e_i$. Consider an arbitrary point $(y_1, \ldots, y_n)$ on $S_1^{\rm fin} \times \cdots S_n^{\rm fin}$. The points $y_i$ are all mutually disjoint and hence the set $\{y_1, \ldots, y_n\}$ has a latest member, and the same holds for sufficiently small neighborhoods $U_0(y_i)$ of the $y_i$. Similarly, for any subset $I \subset \{1, \ldots, n\}$ the infinite tails $S_j^\infty$ and the points $y_i$, $(i, j) \in I \times I^c$,[3] fulfil the hypothesis of Lemma 2.1.7, which states that the $n$ sets $S_j^\infty$ and $\{y_i\}$ have a latest member. The same holds for sufficiently small neighborhoods $U_I(y_i)$ of the points $y_i$, $i \in I$. Let now $U(y_i)$ be the intersection of $U_0(y_i)$ and of all $U_I(y_i)$, where $I$ runs through the subsets of $\{1, \ldots, n\} \setminus \{i\}$. Then for any $I \subset \{1, \ldots, n\}$, the $n$ sets $S_j^\infty$ and $=U(y_i)$, $(i, j) \in I \times I^c$, have a latest member. Of course the same holds for the intersections of these neighborhoods with the corresponding strings,

$$I(y_i) \doteq U(y_i) \cap S_i^{\rm fin}. \tag{2.13}$$

Summarizing, for each $(y_1, \ldots, y_n) \in S_1^{\rm fin} \times \cdots \times S_n^{\rm fin}$ there is a neighborhood $I(y_1) \times \cdots \times I(y_n) \subset S_1^{\rm fin} \times \cdots \times S_1^{\rm fin}$ such that for any $I \subset \{1, \ldots, n\}$ the $n$ sets $S_i^\infty$ and $I(y_j)$, $(i, j) \in I \times I^c$, have a latest member. Now for each $i$ the union

$$\bigcup_{y_i \in S_i^{\rm fin}} I(y_i)$$

is an open covering of the set $S_i^{\rm fin}$. By compactness, there exists a finite sub-covering! That is to say, in the string segment $S_i^{\rm fin}$ there exists a finite number of points $y_i^1, \ldots, y_i^{N_i}$ such that the finite union

$$\bigcup_{\alpha = 1, \ldots N_i} I(y_i^{\alpha_i})$$

still covers $S_i^{\rm fin}$. Of course, these neighborhoods still have a latest member in the sense mentioned after Eq. (2.13). We may assume that the points $y_i^1, \ldots, y_i^{N_i}$ are successive neighbors within $S_i^{\rm fin}$. Then $I(y_i^\alpha)$ and $I(y_i^{\alpha+1})$ have an overlap. Choose, for each $\alpha \in \{1, \ldots, N_i - 1\}$, a number $s_i^\alpha$ such that $x_i + s_i^\alpha e_i$ is contained in the overlap $I(y_i^\alpha) \cap I(y_i^{\alpha+1})$, and define

$$S_i^\alpha \doteq x_i + [s_i^\alpha, s_i^{\alpha+1}]\, e_i.$$

[3] $I^c$ denotes the complement of $I$

Then $S_i^\alpha$ is contained in $I(y_i^\alpha)$, and hence each $n$-tuple of string segments or infinite tails $S_1^{\alpha_1}, \ldots, S_n^{\alpha_n}$ has a latest member, as claimed. This concludes the proof of Prop. 2.2.4. $\square$

# 3 Time-ordered Products of Fields

## 3.1 Considerations on the Point-like Case

In order to study QFT perturbatively, one needs the concept of time ordering of the product of operator-valued fields. Let $\varphi(x)$ and $\varphi(x')$ be two point-like fields, then the time-ordering of the product $\varphi(x')\varphi(x)$ is given by [1]

$$T\varphi(x')\varphi(x) = \begin{cases} \varphi(x')\varphi(x) & \text{if } t' > t \\ \varphi(x)\varphi(x') & \text{if } t > t' \end{cases} , \tag{3.1}$$

where $t$ and $t'$ are the time coordinates of the events $x$ and $x'$, respectively, with respect to a particular frame of reference. If $x'$ and $x$ are time-like or light-like separated all inertial frames agree with respect to their causal order, therefore either $t' > t$ or $t > t'$ unambiguously and equation (3.1) makes perfect sense. However, for space-like separated events their time order is relative, depending on the frame, that would make equation (3.1) ambiguous were it not for the causality axiom which asserts the commutativity of fields for space-like separated events and particularly as a consequence guarantees the validity of (3.1) for $t' = t$ but $x' \neq x$, since in this case $(x' - x)$ is space-like. Thus, the $T-$product is Lorentz invariant and hence well-defined for non-coincidental points. Using definition 2.0.1, we may rewrite (3.1) as

$$T\varphi(x')\varphi(x) = \begin{cases} \varphi(x')\varphi(x) & \text{if } x' \succeq x \\ \varphi(x)\varphi(x') & \text{if } x \succeq x' \end{cases} . \tag{3.2}$$

Using Wick's expansion theorem, we may rewrite (3.2) as

$$T\varphi(x)\varphi(x') = :\varphi(x)\varphi(x'): + \Big(\Omega, T\varphi(x)\varphi(x')\Omega\Big), \tag{3.3}$$

where the double colon stands for the normal order relation and $\Big(\Omega, T\varphi(x)\varphi(x')\Omega\Big)$ is the Feynman propagator associated with $\varphi$. The case in which $x$ coincides with $x'$ is reasonably treated by extending the numerical distribution given by the Feynman propagator in equation (3.3) across the point diagonal $\Delta_2^p \doteq \{(x, x') \in \mathbb{R}^8 : x = x'\}$. This last step is where the subtleties actually lie for both the point-localized and the string-localized fields.

The perturbative construction of interaction models *a la* Epstein and Glaser [13] depends fundamentally on time ordered products of sub-Wick polynomials $W_k$ of the interaction Lagrangean $L$, denoted by $T_n(W_1, W_2, ..., W_n)$. These are operator-valued

[1] We consider the case where the field $\varphi$ is bosonic. In the case it were fermionic, the second line in equation (3.1) would be $-\varphi(x)\varphi(x')$.

distributions on $\mathbb{R}^{4n}$ acting on the domain $\mathcal{D}$ of vectors with finite particle number and smooth momentum space wave functions. Informally, we can write

$$T_n(W_1,\dots,W_n)(f_1,\dots,f_n) =: \int_{(\mathbb{R}^4)^{\times n}} dx_1\cdots dx_n\, T_n\Big(W_1(x_1)\cdots W(x_n)\Big) f_1(x_1)\cdots f_n(x_n).$$

$T_0$ and $T_1$ are given by $T_0 := 1$ and $T_1(W) := W$. Further requirements are:

(p1) (Linearity.) The time-ordered product $T_n$ is an $n$-linear application from the Wick polynomials into operator-valued distributions acting on $\mathcal{D}$.[2]

(p2) (Symmetry.) $T^n(W_1,\dots,W_n)$ is symmetric under permutations $W_i \leftrightarrow W_k$.

(p3) (Causality.) If $x_i \not\precsim x_j$ for all $i \in \{1,\dots,k\}$ and $j \in \{k+1,\dots,n\}$, then the following factorization holds:

$$T_n\left(W_1(x_1)\cdots W_n(x_n)\right) = T_k\left(W_1(x_1)\cdots W_k(x_k)\right) T_{n-k}\left(W_{k+1}(x_{k+1})\cdots W_n(x_n)\right).$$

(p4) (Covariance.) If $W_i$ are *scalar* Wick polynomials, then for all $(a,\Lambda) \in P_+^\uparrow$

$$U(a,\Lambda)\, T_n\left(W_1(x_1)\cdots W_n(x_n)\right) U(a,\Lambda)^{-1} = T_n\left(W_1(a+\Lambda x_1)\cdots W_n(a+\Lambda x_n)\right).$$

(p5) (Wick expansion – scalar field case.) Let $k_1,\dots,k_n \in \mathbb{N}_0$ and let $\mathcal{G}(k_1,\dots,k_n) \doteq \{0,\dots,k_1\}\times\cdots\times\{0,\dots,k_n\}$ be the set of multi-indices $(l_1,\dots,l_n) \in \mathbb{N}_0^{\times n}$ with $0 \le l_i \le k_i$. Then

$$T_n\left(:\varphi^{k_1}(x_1):\cdots:\varphi^{k_n}(x_n):\right) = \sum_{G\in\mathcal{G}(k_1,\dots,k_n)} t_G(x_1,\dots,x_n)\ :\varphi^{l_1}(x_1)\cdots\varphi^{l_n}(x_n):, \tag{3.4}$$

where for $G=(l_1,\dots,l_n)$, $t_G$ is the numerical distribution

$$t_G(x_1,\dots,x_n) = \binom{k_1}{l_1}\cdots\binom{k_n}{l_n}\left(\Omega, T_n\ :\varphi^{k_1-l_1}(x_1):\cdots:\varphi^{k_n-l_n}(x_n):\Omega\right).$$

(The product of distributions in Eq. (3.4) exists due to Epstein-Glaser's "Theorem 0"[13].) The time-ordered products can be constructed inductively: If all $T_k$ are known up to some order $k \le n-1$, then $T_n$ is fixed [13, 14] by (p3) and (p2) outside the small diagonal $\Delta_n$. For a better understanding, let us prove this claim as a theorem:

2 We adopt here the "on-shell formalism".

**Theorem 3.1.1.** *Let $L$ be a particular functional lagrangean and denote by $T_n(x_1,...,x_n) \equiv TL(x_1)\cdots L(x_n)$, $n \in \mathbb{N}$, the family of symmetric operator-valued distributions on $\mathbb{R}^{4n}$ satisfying the causality property (p3). Then the equation*

$$T_n(x_1,...,x_n) = T_k(x_1,...,x_k)\, T_{n-k}(x_{k+1},...,x_n)\,. \tag{3.5}$$

*fixes $T_n$ inductively up to the total diagonal $\Delta_n \doteq \{(x_1,...,x_n) : x_1 = \cdots = x_n\}$ in the following sense: If $T_k$ is known for every $k<n$, then $T_n$ is fixed on $\mathbb{R}^n \backslash \Delta_n$ .*

*Proof.* Suppose $(x_1,...,x_n) \notin \Delta_n$, then there are at least two points in the set $\{x_1,x_2,...,x_n\}$ and hence there exists a partition $P = \{I, I^c\}$ of the set $\{1,...,n\}$ such that $x_i \succeq x_{i'}$ $\forall i \in I$ and $\forall i' \in I^c$. Thus, the causality property (p3) implies

$$T_n(\{1,...,n\}) = T(I)\, T(I^c) \tag{3.6}$$

where $T(I) := T(x_{i_1},...,x_{i_k})$, $I = \{i_1,...,i_k\}$ and $k = |I|$. If the points $x_i$ are all causally connected, that is $x_i^2 \geq 0$ $\forall i \in \{1,...,n\}$, then $I$ is uniquely defined ($x_{i_1} \succeq x_{i_2} \succeq \cdots \succeq x_{i_n}$ for all reference frames, where $\{i_1,...,i_n\} = \{1,...,n\}$) and we are done. If, on the other hand, $\exists l \in \{1,...,n\}$ such that $x_l^2 < 0$, (that is, $x_l$ is space-like) then $I$ is not uniquely defined, and in this case we must prove the independence of equation (3.6) on the choice of the proper subset $I$. With that purpose in mind, let $Q = \{J, J^c\}$ be another partition of the set $\{1,...,n\}$ such that $x_j \succeq x_{j'}$ $\forall j \in J$ and $\forall j' \in J^c$. Using the identities $I = I \cap J \,\dot\cup\, I \cap J^c$ and $I^c = I^c \cap J \,\dot\cup\, I^c \cap J^c$, we can rewrite the right-hand side of equation (3.6) as

$$T(I)\, T(I^c) = T\left(I \cap J \,\dot\cup\, I \cap J^c\right) T\left(I^c \cap J \,\dot\cup\, I^c \cap J^c\right). \tag{3.7}$$

It is easy to see that $I \cap J \,\dot\succeq\, I \cap J^c$ and $I^c \cap J \,\dot\succeq\, I^c \cap J^c$ ( in the sense of the elements of the corresponding sets) and so (3.7) becomes

$$T(I)\, T(I^c) = T(I \cap J)\, T(I \cap J^c)\, T(I^c \cap J)\, T(I^c \cap J^c)\,. \tag{3.8}$$

Analagously, for the sets $J$ and $J^c$ we have the following identities $J = J \cap I \,\dot\cup\, J \cap I^c$ and $J^c = J^c \cap I \,\dot\cup\, J^c \cap I^c$, yield

$$T(J)\, T(J^c) = T\left(J \cap I \,\dot\cup\, J \cap I^c\right) T\left(J^c \cap I \,\dot\cup\, J^c \cap I^c\right), \tag{3.9}$$

and since both $J \cap I \,\dot\succeq\, J \cap I^c$ and $J^c \cap I \,\dot\succeq\, J^c \cap I^c$, we have

$$T(J)\, T(J^c) = T(J \cap I)\, T(J \cap I^c)\, T(J^c \cap I)\, T(J^c \cap I^c)\,. \tag{3.10}$$

It is clear that the arguments of the second and third terms in equation (3.10) are such that $J \cap I^c \succeq J^c \cap I$ from the perspective of the partition $Q$. However, from the perspective of the partition $P$ we also have $J^c \cap I \succeq J \cap I^c$. This means that the points in $J^c \cap I$

are causally disjoint from the points in $J \cap I^c$, which enables us to commute the terms $T(J \cap I^c)$ and $T(J^c \cap I)$ in (3.10) obtaining as a result

$$T(J)\,T(J^c) = T(J \cap I)\,T(J^c \cap I)\,T(J \cap I^c)\,T(J^c \cap I^c) = T(I)\,T(I^c) \tag{3.11}$$

□

From the previous theorem we know that the time-ordering of an $n$-fold product of any point-like interaction lagrangean $L(x)$ is well-defined everywhere, except in the total (small) diagonal $\Delta_n$.

The "UV problem" of the divergencies consists, in this context, in the extension of the $T$-*product* across $\Delta_n$. This extension is not unique in general, and the choice of possible extensions is restricted by the requirements (p4) and (p5), which therefore may be called (re-) normalization conditions. By (p4), the $T_n$ are fixed up to the origin in $\mathbb{R}^{4n}$. The condition (p5) holds due to Wick's theorem outside the union of all diagonals, i.e., whenever $x_i \neq x_j$, and the requirement that it holds on all $\mathbb{R}^{4n}$ is a normalization condition by virtue of which the extension problem needs to be considered only for the numerical distributions $t_G$.

Let us now investigate, as an example, the renormalizability of a scalar theory whose interaction lagrangean is of the form $L(x) =: \varphi^m(x) :$. In this case, the Wick expansion (3.4) for $n$ vertices is given by

$$TL(x_1)\cdots L(x_n) = \sum_{G \in \mathcal{G}(m,\ldots,m)} t_G(x_1,\ldots,x_n)\ :\varphi^{b_1}(x_1)\cdots\varphi^{b_n}(x_n):, \tag{3.12}$$

where for $G = (b_1,\ldots,b_n)$, $t_G$ is the numerical distribution

$$t_G(x_1,\ldots,x_n) = \binom{m}{b_1}\cdots\binom{m}{b_n}\left(\Omega, T_n\ :\varphi^{m-b_1}(x_1):\cdots:\varphi^{m-b_n}(x_n):\Omega\right). \tag{3.13}$$

Applying Wick's expansion to both sides of equation (3.6) we get

$$\begin{aligned}
&\sum_{G \in \mathcal{G}_n} t_G(x_1,\cdots,x_n) : \varphi^{b_1}(x_1)\cdots\varphi^{b_n}(x_n) : \\
&= \sum_{G_1 \in \mathcal{G}_k}\sum_{G_2 \in \mathcal{G}_{n-k}} t_{G_1}(I)t_{G_2}(I^c) : \varphi^{b_{i_1}}(x_{i_1})\cdots\varphi^{b_{i_k}}(x_{i_k}) :: \varphi^{b_{i_{k+1}}}(x_{i_{k+1}})\cdots\varphi^{b_{i_n}}(x_{i_n}) : \\
&= \sum_{G_1,G_2} t_{G_1}(I)t_{G_2}(I^c) : \varphi^{b_I}(I) :: \varphi^{b_{I^c}}(I^c) :
\end{aligned} \tag{3.14}$$

where $\mathcal{G}_n$ is the set of graphs with $n$ vertices $x_1,\cdots,x_n$ with $b_i$ external lines coming out of the vertex $x_i$. The set $\{I, I^c\}$ is a partition of the set $\{x_1,\cdots,x_n\}$ with $I = \{x_{i_1},\cdots,x_{i_k}\}$ and $I^c = \{x_1,\cdots,x_n\}\backslash I = \{x_{i_{k+1}},\cdots,x_{i_n}\}$. Also, we have defined $\varphi^{b_{i_1}}(x_{i_1})\cdots\varphi^{b_{i_k}}(x_{i_k}) \stackrel{def}{=}$

$\varphi^{b_I}(I)$, where $b_I = b_{i_1} + \cdots + b_{i_k}$ and $b_{I^c} = b_{i_{k+1}} + \cdots b_{i_n}$ are the total number of external lines coming out of the $I$ and $I^c$ vertices, respectively, in the graph $G$.

Using Wick's expansion for the product of the normal ordered terms in equation (3.14) we get

$$\begin{aligned}
&\sum_G t_G(x_1, \cdots, x_n) : \varphi^{b_1}(x_1) \cdots \varphi^{b_n}(x_n) : \\
&= \sum_{G_0, G_1, G_2} t_{G_1}(I) t_{G_2}(I^c) \prod_{(i,j) \in I \times I^c} [\Delta_F(x_i - x_j)]^{l_{ij}} : \varphi^{b'_I}(I) \varphi^{b'_{I^c}}(I^c) :
\end{aligned} \tag{3.15}$$

where $b'_I$ and $b'_{I^c}$ are the number of external lines in the graphs $G_1$ and $G_2$, respectively. In addition, $G_0$ runs through the graphs whose internal lines connect only vertices in the same set $I$ or $I^c$, and $b := b_I + b_{I^c} = b'_I + b'_{I^c} - 2l_{I,I^c}$, where $l_{I,I^c} = \sum l_{ij}$ is the number of lines connecting vertices in $I$ with vertices in $I^c$, as ilustrated in figures 4 and 5. We may now calculate the scaling degree (definition B.1.3 in appendix B.1) of the distribution $t_G$ by induction:

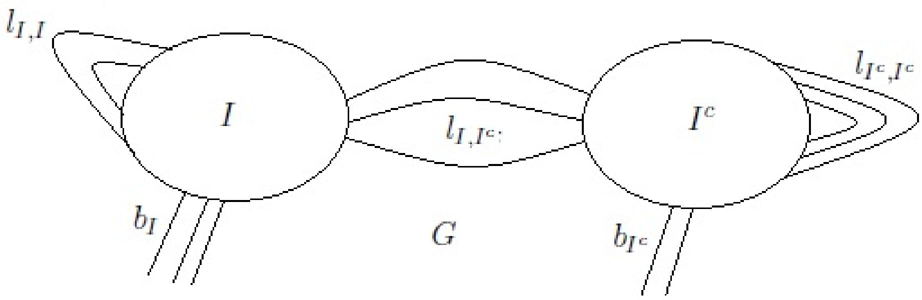


**Figure 4** – Graph $G$ with $n$ vertices $\{x_1, \cdots, x_n\} = I \cup I^c$ and $b_I + b_{I^c}$ external lines.

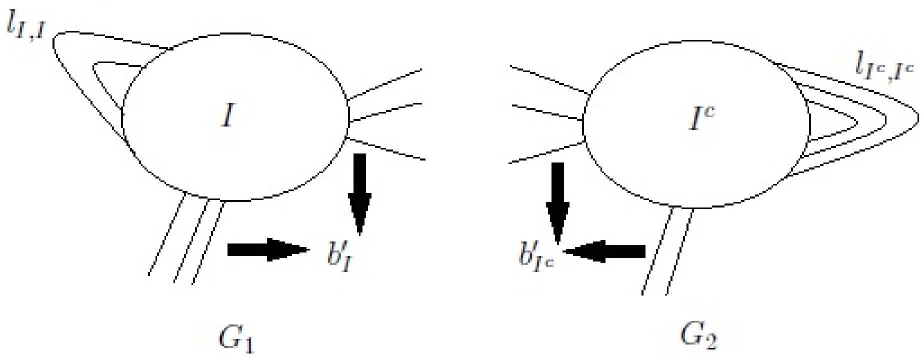


**Figure 5** – Dichotomy of the graph $G$ into $G_1$ and $G_2$, where graph $G_1$ has $k = |I|$ vertices and $b'_I$ external lines and graph $G_2$ has $n - k = |I^c|$ vertices and $b'_{I^c}$ external lines.

**Theorem 3.1.2.** *For $G \in \mathcal{G}_n^b$, that is, $G$ has $n$ vertices and $b$ external lines, the distribution $t_G$ defined in (3.13) has scaling degree $mn - b$.*

*Proof.* From equation (3.15) we get

$$\mathrm{sd}(t_G) = \mathrm{sd}(t_{G_1}) + \mathrm{sd}(t_{G_2}) + \mathrm{sd}(\Delta_F) \sum l_{ij} \tag{3.16}$$

where we have used the properties of scaling degree stated in B.1.8. The graph $G_1$ has $k$ vertices and $b'_I$ external lines whilst $G_2$ has $n-k$ vertices and $b'_{I^c}$ external lines. Hence, using the induction hypothesis for $t_{G_1}$ and $t_{G_2}$ and the result $sd(\Delta_F) = 2$ (obtained in example B.1.7 in appendix B.1), we get

$$\begin{aligned}\mathrm{sd}(t_G) &= [mk - b'_I] + [m(n-k) - b'_{I^c}] + 2l_{I,I^c} && (3.17)\\ &= mn - (b'_I + b'_{I^c}) + 2l_{I,I^c} && (3.18)\\ &= mn - b && (3.19)\end{aligned}$$

To prove the basis step, note that for $n = 1$ we have $t_G(x_1) = 1$ with scaling degree 0 and $b = m$, hence $\mathrm{sd}(t_G) = 0 = m - b$. This completes the proof. □

We may now calculate the degree of divergence[3] of the distribution $t_G$ by

$$\mathrm{div}(t_G) := \omega = \mathrm{sd}(t_G) - \mathrm{codim}(\Delta_n) \tag{3.20}$$

where $\mathrm{codim}(\Delta_n) = \dim(\mathbb{R}^{4n}) - \dim(\Delta_n) = 4(n-1)$. Therefore,

$$\omega = (m-4)n + 4 - b \tag{3.21}$$

We may now analyze the renormalizability of a scalar theory with interaction lagrangean of the form $L(x) =: \varphi^m(x) :$. By inspecting equation (3.21) we notice that if $m > 4$, the degree of divergence will increase with the number of vertices no matter how large the number of external lines is, which makes the theory non-renormalizable in this case. As an example, for $m = 4$ we have $\omega = 4 - b$ and only graphs with $b = 2$ and $b = 4$ will be superficially divergent. Therefore, the theory is renormalizable, since only a finite number of physical constants has to be redefined. The redefinition of physical parameters in standard renormalization schemes corresponds to the non-unique extension of distributions in the Epstein-Glaser scheme. As stated in theorem B.1.9 in appendix B.1, for $\omega \geq 0$ the extension of the distribution through the origin is unique up to derivatives of the delta distribution and each term contains an undefined constant which can be fixed by the imposition of physical principles, such as conservation of energy and momentum. The larger $\omega$ is, the larger will be the number of parameters to be fixed. Furthermore, if $\omega < 0$ the extension is unique.

Having constructed the *T-product*, the Bogoliubov's S-matrix is defined as a functional associating with a test function $g \in \mathcal{D}(\mathbb{R}^4)$ and an interaction Lagrangean $L$

[3] See the definition of degree of divergence in the last paragraph of appendix B.1.

the following series

$$
\begin{aligned}
S(gL) &\doteq \sum_{n=0}^{\infty} \frac{i^n}{n!} T_n(L, L, ..., L)(g, g, ..., g) \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n g(x_1) \cdots g(x_n) T_n(L(x_1), \cdots, L(x_n)),
\end{aligned}
\tag{3.22}
$$

in the formal sense, i.e. without requiring the convergence of the series. In the so called adiabatic limit, $g(x) \to g = constant$, equation (3.22) becomes the S-matrix of the model[4].

## 3.2 The String-like Case

### 3.2.1 Introduction

Let us briefly consider the case of the Proca field $A^p_\mu(x)$, which describes massive particles with spin 1. The Proca field is divergence free, and hence satisfies the Proca equation given by

$$\partial_\mu F^{\mu\nu}(x) + m^2 A^{p\nu}(x) = 0, \tag{3.23}$$

where $F^{\mu\nu}$ is the field strength,

$$F_{\mu\nu}(x) \doteq \partial_\mu A^p_\nu(x) - \partial_\nu A^p_\mu(x). \tag{3.24}$$

It can be constructed [4] a string-localized version of the Proca field $A_\mu(x, e)$ with the same field strength as its point-localized counterpart, that is

$$\partial_\mu A_\nu(x, e) - \partial_\nu A_\mu(x, e) = F_{\mu\nu}(x). \tag{3.25}$$

By equation (3.25) we can assert that $A^p_\mu(x)$ and $A_\mu(x, e)$ differ only by the gradient of a scalar field $\phi(x, e)$, called the **escort field**, that is

$$A_\mu(x, e) = A^{\mathrm{p}}_\mu(x) + \partial_\mu \phi(x, e), \tag{3.26}$$

where we may represent the fields $A_\mu(x, e)$ and $\phi(x, e)$ as line integrals as follows

$$A_\mu(x, e) = \int_0^\infty ds F_{\mu\nu}(x + se)e^\nu \tag{3.27}$$

and

$$\phi(x, e) = \int_0^\infty ds A^{\mathrm{p}}_\nu(x + se)e^\nu. \tag{3.28}$$

The interesting point is that, although the point-localized Proca field has scaling dimension 2, and thus a bad UV behaviour, its string-localized version has scaling dimension[5] 1 (For the two-point functions of the fields $\phi(x, e), A^{\mathrm{p}}_\mu(x)$and$A_\mu(x, e)$, see Appendix B.3).

[4] We are not going to give the proof of renormalizability for the point-localized QED. For detailed treatments along the lines of Epstein and Glaser causal method see [4, 13, 14].

[5] See definition B.1.3 in appendix A.

The particle types in massive QED are the “massive photon”, the electron and the positron, and the coupling is described by the interaction Lagrangean

$$L^{\mathrm{p}}(x) \doteq j^{\mu}(x) A^{\mathrm{p}}_{\mu}(x), \tag{3.29}$$

where $j^{\mu}(x) \doteq :\bar{\psi}(x)\gamma^{\mu}\psi(x):$ is the current operator and $\psi$ is the free Dirac field. Now the scaling dimension of $j^{\mu}$ is three and that of $A^{\mathrm{p}}$ is two, hence that of $L^{\mathrm{p}}$ is five. Thus the model is non-renormalizable as it stands. Our way out, analogous to the BRST approach [24], is to consider its string-localized version

$$L^{\mathrm{s}}(x,e) \doteq j^{\mu}(x) A_{\mu}(x,e), \tag{3.30}$$

which has a better scaling dimension, namely four[6]. By Eq. (3.26) and current conservation, $\partial_{\mu} j^{\mu} = 0$, the two interaction Lagrangeans differ by the divergence of the string-localized vector field $V^{\mu}(x,e) \doteq j^{\mu}(x)\phi(x,e)$, where $\phi$ is the escort field:

$$L^{\mathrm{p}}(x) = L^{\mathrm{s}}(x,e) - \partial_{\mu} V^{\mu}(x,e) \tag{3.31}$$

where $\partial_{\mu}$ is the partial derivative with respect to $x$.

### 3.2.2 Time-ordering of Linear Factors

We set out to define the time-ordered products of linear string-localized fields $\varphi$,

$$T\varphi(x_1,e_1)\cdots\varphi(x_n,e_n) \;\doteq\; T_n(x_1,e_1,\ldots,x_n,e_n). \tag{3.32}$$

These are operator-valued distributions on $(\mathbb{R}^4 \times H)^{\times n}$ acting on the domain $\mathcal{D}$ of vectors with finite particle number and smooth momentum space wave functions, which are required to share the following properties.

(P1) $T_1$ is given by $T_1(x,e) := \varphi(x,e)$.

(P2) (Linearity.) The $T$-product is an $n$-linear application from the space of linear field operators $\varphi(x,e)$ into operator-valued distributions acting on $\mathcal{D}$.

(P3) (Symmetry.) $T_n(x_1,e_1,\ldots x_n,e_n)$ is symmetric under permutations of the joint variables $(x_i,e_i)$.

(P4) (Causality.) If $S_{x_i,e_i} \succcurlyeq S_{x_j,e_j}$ for all $i \in \{1,\ldots,k\}$ and $j \in \{k+1,\ldots,n\}$, then the following factorization holds:

$$T_n(x_1,e_1,\ldots,x_n,e_n) = T_k(x_1,e_1,\ldots,x_k,e_k)\, T_{n-k}(x_{k+1},e_{k+1},\ldots,x_n,e_n).$$

[6] The same effect of constructing a Lagrangean with scaling dimension equal to four can be obtained by the standard description of point-like fields and gauge invariance. However, in this case the field is unphysical since it is an element of an indefinite inner product space, the so called *Krein space*.

Before we turn to the construction of the $T$ products, we recall Wick's theorem for linear fields, which is also valid in the string-localized case ($\varphi(i)$ denotes $\varphi(x_i)$ or $\varphi(x_i, e_i)$ in the string-localized case):

$$\varphi(1)\cdots\varphi(n) = \sum_G \prod_{l\in E_{\rm int}} \Big(\Omega, \varphi(r(l))\varphi(s(l))\Omega\Big) : \prod_{l\in E_{\rm ext}} \varphi(s(l)): . \tag{3.33}$$

Here, $G$ runs through the set of all graphs with vertices $\{1,\ldots,n\}$ and oriented lines, such that from every vertex emanates one line. The lines either connect two vertices (internal lines, $l \in E_{\rm int}$) or go from a vertex to the exerior (external lines, $l \in E_{\rm ext}$). The initial vertex of an internal line $l$ (source $s(l)$) has a smaller index than its final vertex (range $r(l)$). The external lines only have sources.

Let us recall how the time-ordered products are constructed in the point-local case. In a first step, one shows that Wick's expansion (3.33) also holds for the time-ordered products outside the large diagonal $\{x_i \neq x_j\}$, namely:

$$T\,\varphi(1)\cdots\varphi(n) = \sum_G \prod_{l\in E_{\rm int}} \Big(\Omega, T\varphi(r(l))\varphi(s(l))\Omega\Big) : \prod_{l\in E_{\rm ext}} \varphi(s(l)): . \tag{3.34}$$

(The vacuum expectation value $\Big(\Omega, T\,\varphi(x)\varphi(y)\Omega\Big)$ is called the *Feynman propagator*.) This is shown by induction, using the fact that $n$ distinct points always have a latest member in the sense of $\succcurlyeq$. In a second step, one constructs the extension across the large diagonal (requiring certain (re-) normalization conditions). If the scaling degree of the Feynman propagator is smaller than 4, then the $T$ products are fixed (on all $\mathbb{R}^{4n}$), namely, they are given by Eq. (3.34). On the other hand, if the scaling degree of the Feynman propagator is $\geq 4$ one may add, depending on the scaling degree and the number of internal vertices of the graph in (3.34), renormalization terms in the form of delta distributions (and derivatives) in the difference variables with "internal" indices. This is the case for fields with spin $\geq 1$ acting in a Hilbert space.

We show here that for string-localized fields $\varphi(x,e)$ the $T_n$ are fixed outside the large string diagonal $\Delta_n$ just by the geometric time-ordering prescription, namely they are given by the same expression (3.34) as in the point-like case. As mentioned in the introduction, the problem we have to overcome is the fact that the set of points in $(\mathbb{R}^4 \times H)^{\times n}$ which correspond to strings that are *not* comparable in the sense of $\succcurlyeq$ is much larger than $\Delta_n$, in fact it contains an open set. We use our results on string chopping from the last section to show that they are nevertheless fixed outside $\Delta_n$. Recall that we are dealing with string-localized fields that can be written as line integrals over point-localized fields as in Eq. (6). Thus, for any chopping of the string $S_{x,e} = \cup_\alpha S^\alpha$ as in Eq. (2.9), the field $\varphi(x,e)$ can be written as a sum

$$\varphi(x,e) = \sum_{\alpha=1}^{N+1} \varphi^\alpha(x,e)\,, \qquad \text{where} \qquad \varphi^\alpha(x,e) \doteq \int_{s_{\alpha-1}}^{s_\alpha} ds\, u(s)\,\varphi_{\rm p}(x+se) \tag{3.35}$$

is localized on the segment $S^\alpha$. (We put $s_0 \doteq 0$ and $s_{N+1} \doteq \infty$.)

We start with two fields. If the two strings $S_{x,e} \doteq S$ and $S_{x',e'} \doteq S'$ are comparable, then (P4) implies that

$$T\varphi(x,e)\varphi(x',e') = \begin{cases} \varphi(x,e)\varphi(x',e') & \text{if } S \succcurlyeq S' \\ \varphi(x',e')\varphi(x,e) & \text{if } S \preccurlyeq S' \end{cases} . \tag{3.36}$$

This is well-defined, for if $S$ is both later and earlier than $S'$ then it is space-like separated from $S'$ by Lemma 2.1.1 and the fields commute, so that both lines in (3.36) are valid. The problem is that there is an open set of pairs of strings which are not comparable, namely whenever $S$ meets both the past and the future of $S'$. This is solved by the concept of string chopping, which fixes the $T$ product outside the string diagonal:

**Proposition 3.2.1.** *The time ordered product $T\varphi(x,e)\varphi(x'e')$ is uniquely fixed outside the string-diagonal $\Delta_2$ by (P1) through (P4). It satisfies Wick's expansion*

$$T\varphi(x,e)\varphi(x',e') = :\varphi(x,e)\varphi(x',e'): + \Big(\Omega, T\varphi(x,e)\varphi(x',e')\Omega\Big). \tag{3.37}$$

*Proof.* If the two strings $S_{x,e} \doteq S$ and $S_{x',e'} \doteq S'$ are comparable, their $T$ product has been defined in Eq. (3.36). If the strings are not comparable, then we cut one string, say $S$, into two pieces $S = S^1 \cup S^2$ such that the pairs $(S^1, S')$ and $(S^2, S')$ are comparable (see Prop. 2.2.1). As explained in Eq. (3.35), the field $\varphi(x,e)$ can be written as a sum $\varphi = \varphi^1 + \varphi^2$, where the field $\varphi^\alpha$ is localized on $S^\alpha$, $\alpha = 1, 2$. By linearity (P2) of the $T$ product, we have

$$T\varphi(x,e)\varphi(x',e') = T\varphi^1(x,e)\varphi(x',e') + T\varphi^2(x,e)\varphi(x',e'), \tag{3.38}$$

where both terms are fixed as in Eq. (3.36). We need to show independence of the chosen chopping. Given a different chopping $S = \tilde S^1 \cup \tilde S^2$, one of the new pieces $\tilde S^\alpha$ is contained in one of the old pieces $S^\beta$. We may assume that $\tilde S^1 \subset S^1$. Then we have

$$S^1 = \tilde S^1 \cup S^{12}, \quad \tilde S^2 = S^2 \cup S^{12}, \tag{3.39}$$

where $S^{12} \doteq S^1 \setminus \tilde S^1$ is the "middle piece". The field decomposes as $\varphi = \tilde\varphi^1 + \tilde\varphi^2$, where the operator $\tilde\varphi^\alpha$ is localized on $\tilde S^\alpha$, $\alpha = 1, 2$, and by Eq. (3.39) we have

$$\tilde\varphi^2(x,e) = \varphi^{12}(x,e) + \varphi^2(x,e) \quad \text{and} \quad \tilde\varphi^1(x,e) + \varphi^{12}(x,e) = \varphi^1(x,e),$$

where $\varphi^{12}(x,e)$ is localized on the middle piece $S^{12}$. With respect to the new chopping, we therefore have

$$\begin{aligned} T\varphi(x,e)\varphi(x',e') &= T\tilde\varphi^1(x,e)\varphi(x',e') + T\tilde\varphi^2(x,e)\varphi(x',e') \\ &= T\tilde\varphi^1(x,e)\varphi(x',e') + T\varphi^{12}(x,e)\varphi(x',e') + T\varphi^2(x,e)\varphi(x',e') \\ &= T\varphi^1(x,e)\varphi(x',e') + T\varphi^2(x,e)\varphi(x',e'). \end{aligned}$$

This proves independence of the chosen chopping in Eq. (3.38), and we have shown uniqueness outside $\Delta_n$. Substituting Eq. (3.36) into Eq. (3.38) and applying Wick's theorem for ordinary products, yields Wick's expansion (3.37) for the $T$ products. $\square$

We turn to the case of $n > 2$ fields, and show that Wick's expansion (3.34) also holds for string-localized fields – outside the large string diagonal:

**Proposition 3.2.2.** *The time-ordered n-fold product of a string-localized free field $\varphi(x_i, e_i)$ is uniquely fixed outside the the large string diagonal $\Delta_n$, namely there holds*

$$T\varphi(x_1, e_1) \cdots \varphi(x_n, e_n) = \sum_G \prod_{l\in E_{int}} \Big(\Omega, T\varphi(x_{s(l)}, e_{s(l)})\varphi(x_{r(l)}, e_{r(l)})\Omega\Big) : \prod_{l\in E_{ext}} \varphi(x_{s(l)}, e_{s(l)}): \quad (3.40)$$

*outside the large string diagonal. (Same notation as above.)*

*Proof.* Let $(x_0, e_0, \ldots, x_n, e_n)$ be outside the large string diagonal. That means that the strings $S_i \doteq S_{x_i,e_i}$ are mutually disjoint, $i = 0, \ldots, n$. We wish to determine $T_{n+1} \doteq T\varphi(0)\cdots\varphi(n)$, where we have written $\varphi(i) \doteq \varphi(x_i, e_i)$, under the induction hypothesis that the formula (3.40) is valid for $T_n = T\varphi(1)\cdots\varphi(n)$. Choose a chopping of the $n+1$ strings as in Prop. 2.2.4, and let $\varphi(i) = \sum_{\alpha=1}^{N_i+1} \varphi^\alpha(i)$ be the corresponding decomposition as in Eq. (3.35). Then by linearity (P2),

$$T_{n+1} = \sum_{\alpha_0,\ldots,\alpha_n} T\varphi^{\alpha_0}(0)\cdots\varphi^{\alpha_n}(n).$$

For given $(\alpha_0, \ldots, \alpha_n)$, denote by $i_0$ the index of the latest member of the set of string segments $\{S_0^{\alpha_0}, \ldots, S_n^{\alpha_n}\}$ as in Prop. 2.2.4. Then by (P3) and (P4),

$$T_{n+1} = \sum_{\alpha_0,\ldots,\alpha_n} \varphi^{\alpha_{i_0}}(i_0)\, T\prod_{i\in I} \varphi^{\alpha_i}(i),$$

where we have written $I \doteq \{0, \ldots, n\} \setminus \{i_0\}$. By the induction hypothesis, this is

$$\sum_{\alpha_0,\ldots,\alpha_n} \varphi^{\alpha_{i_0}}(i_0) \sum_G \prod_{l\in E_{\rm int}} \langle T\varphi^{\alpha_{s(l)}}(s(l))\varphi^{\alpha_{r(l)}}(r(l))\rangle : \prod_{l\in E_{\rm ext}} \varphi^{\alpha_{s(l)}}(s(l)):,$$

where $G$ runs through all graphs $\mathcal{G}(I)$ with vertices $I$, and $\langle\cdot\rangle$ denotes the vacuum expectation value. Using Wick's Theorem for ordinary products, we have

$$\varphi^{\alpha_{i_0}}(i_0) : \prod_{i\in I_{\rm ext}} \varphi^{\alpha_i}(i): = \\ :\varphi^{\alpha_{i_0}}(i_0) \prod_{i\in I_{\rm ext}} \varphi^{\alpha_i}(i): + \sum_{i\in I_{\rm ext}} \langle \varphi^{\alpha_{i_0}}(i_0)\varphi^{\alpha_i}(i)\rangle :\varphi^{\alpha_{i_0}}(i_0) \prod_{j\in I_{\rm ext}\setminus\{i\}} \varphi^{\alpha_j}(j):,$$

where $I_{\rm ext}$ denotes the set of vertices with external lines, $\{s(l), l \in E_{\rm ext}\}$. Now since $i_0$ is the latest member of the string segments, we may replace the vacuum expectation value by the time-ordered one, $\langle T\varphi^{\alpha_{i_0}}(i_0)\varphi^{\alpha_i}(i)\rangle$. We arrive at

$$T_{n+1} = \sum_{\alpha_0,\ldots,\alpha_n} \sum_{G'} \prod_{l\in E'_{\rm int}} \langle T\varphi^{\alpha_{s(l)}}(s(l))\varphi^{\alpha_{r(l)}}(r(l))\rangle : \prod_{l\in E'_{\rm ext}} \varphi^{\alpha_{s(l)}}(s(l)):,$$

where $G'$ runs through all graphs with vertices $\{0,\dots,n\}$, internal lines $E'_{\rm int}$ and external lines $E'_{\rm ext}$. Now the index $i_0$ (which depends on the tupel $\underline{\alpha}$) is not discriminated any more, and we can perform the sum over $\alpha$'s:

$$\begin{aligned} T_{n+1} &= \sum_{G'} \prod_{l\in E'_{\rm int}} \langle T \sum_{\alpha_{s(l)}} \varphi^{\alpha_{s(l)}}(s(l)) \sum_{\alpha_{r(l)}} \varphi^{\alpha_{r(l)}}(r(l))\rangle \ : \prod_{l\in E'_{\rm ext}} \sum_{\alpha_{s(l)}} \varphi^{\alpha_{s(l)}}(s(l)): \\ &= \sum_{G'} \prod_{l\in E'_{\rm int}} \langle T\varphi(s(l))\varphi(r(l))\rangle \ : \prod_{l\in E'_{\rm ext}} \varphi(s(l)): \end{aligned}$$

This is just the claimed equation (3.40). □

An extension of the time-ordered product across the large string-diagonal is not defined up to this point. To fix it, one extends first the Feynman propagator across $\Delta_2$. A basic (re-) normalization condition is that the scaling degree may not be increased. One valid extension consists in replacing $\delta(p^2-m^2)\theta(p_0)$ by $i/[2\pi(p^2-m^2+i\varepsilon)]$ in the Fourier transform of the two-point function. The question if other extensions are permitted depends on the scaling degrees of the Feynman propagator with respect to the various submanifolds of $\Delta_2$ and their respective co-dimensions. We consider an example in Appendix B.2. In a second step, one can define the time-orderd product by Wick's expansion (3.40). This would amount to requiring Wick's expansion as a further normalization condition.

### 3.2.3 Final Comments

We have constructed (outside $\Delta_n$) the time ordered products of string-localized linear fields, but not of Wick polinomials. The construction of the latter runs into the following problem. For simplicity we consider a Wick monomial of the form

$$W(x,e) \doteq \,:\chi(x)\varphi(x,e):, \tag{3.41}$$

where $\chi$ is a point-localized field with non-vanishing two-point function with $\varphi$. (For example, $\chi=\varphi_{\rm p}$ from Eq. (6).) We wish to tell just from the requirements (P2), (P3) and (P4) who $TW(x,e)W(x',e')$ is, if the strings $S\doteq S_{x,e}$ and $S'\doteq S_{x',e'}$ do not intersect, yet are not comparable. A typical case is when one string, say $S$, emanates from the causal future of $S'$ and ends up in its causal past. The best we can do is to cut $S$ into two pieces $S=S^1\cup S^2$ as in Prop. 2.2.1 such that $S^1$ is the finite segment (containing the point $x$) and $S^1 \succcurlyeq S'$, while $S^2$ is the infinite tail and $S^2 \preccurlyeq S'$. ($S^2$ is of the form $S^2=x+[s_0,\infty)e$.) Then, as in Eq. (3.35), $W$ is a sum $W=W^1+W^2$, where in particular

$$W^2(x,e) = \int_{s_0}^{\infty} ds\, u(s) :\chi(x)\varphi_{\rm p}(x+se): .$$

Now the problem is that $W^2$ is, due to the factor $\chi(x)$, *not* localized on $S^2$ — rather, it is "bi-localized" on $\{x\}\cup S^2$! Note that $S^2$ is earlier than $S'$ but $x$ is *not*, since it is in $\overline{V_+(S')}$. Therefore, $TW^2(x,e)W(x',e')$ is not fixed by (P4), in particular it does not factorize as

$W(x',e')W^2(x,e)$ even though $S' \not\succcurlyeq S^2$. Similar considerations hold for more general Wick monomials of the form $:\chi(x)^l\varphi^k(x,e):$.

We conclude that, in contrast to the linear case, the time-ordered products of Wick monomials are fixed by the axioms (P2) through (P4) only outside an open set, namely the set of pairs of strings which are incomparable. The extension into this set requires an infinity of parameters: It cannot be fixed by a finite set of normalization conditions.

We conjecture that this problem can be solved as follows. Recall from the discussion in the introduction chapter that in the construction of interacting models one has to start from an interaction Lagrangean that differs from some point-localized Lagrangean by a divergence. For the point-localized Lagrangean $L^{\mathrm{p}}$ there holds the strong form of Wick's expansion outside the large (point-) diagonal, which fixes the products $TL^{\mathrm{p}}\cdots L^{\mathrm{p}}$ through the Feynman propagators. We conjecture that the required string independence condition[7] (equivalence of the string- and point-localized Lagrangeans) implies that the same expansion holds for the string-localized Lagrangean outside $\Delta_n$ (where it is well-defined). From here, one would have to extend the product of Feynman propagators in various steps across $\Delta_n$. In models like massive QED, where the interaction Lagrangean $j^\mu(x)A_\mu(x,e)$ is linear in the string-localized field $A_\mu$, we conjecture that the SI-condition fixes the extension outside the large *point* diagonal. The question of renormalizability then amounts to the question if the complete extension is fixed by a finite number of parameters, which does not increase with the order $n$. This is work in progress.

[7] Note that Eq. (3.31) implies that the product $L^{\mathrm{p}}(x_1)\cdots L^{\mathrm{p}}(x_n)$ differs from the $n$-fold product of $L^{\mathrm{s}}$ by derivative terms containing the $V_\mu$. We require that this fact survives the time-ordering, in other words: that the time ordering of the $T$-products $T_nL^{\mathrm{s}}\cdots L^{\mathrm{s}}V^\mu\cdots V^\nu$ can be defined so that "the derivatives can be taken out of the $T$-products":

$$TL^{\mathrm{p}}{}_1\cdots L^{\mathrm{p}}{}_n \stackrel{!}{=} TL^{\mathrm{s}}{}_1\cdots L^{\mathrm{s}}{}_n + \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|}\,\partial_{\mu_1}\cdots\partial_{\mu_k} TV^{\mu_1}_{i_1}\cdots V^{\mu_k}_{i_k} L^{\mathrm{s}}{}_{j_1}\cdots L^{\mathrm{s}}{}_{j_{n-k}}. \tag{3.42}$$

(Here we have written $I=\{i_1,\dots,i_k\}$, $I^c=\{j_1,\dots,j_{n-k}\}$, $\partial_{\mu_i}=\frac{\partial}{\partial x_i^{\mu_i}}$, and $W_i=W(x_i,e_i)$ for $W=L^{\mathrm{p}}$, $L^{\mathrm{s}}$ or $V^\mu$.) This is a (re-) normalization condition for the $T$-products $T_nL^{\mathrm{s}}\cdots L^{\mathrm{s}}V^\mu\cdots V^\nu$, which we call *perturbative string-independence*. (The condition can be formulated without mentioning the $L^{\mathrm{p}}$, namely: the right hand side of Eq. (3.42) be independent of the $e$'s.) It is analogous to the condition of "perturbative gauge invariance" in [24]. If it can be satisfied for all $n$, then the Bogoliubov S-matrix for $L^{\mathrm{s}}$ coincides with that of $L^{\mathrm{p}}$ in the adiabatic limit, $g\to$ const., since the boundary terms vanish. In [25] it has been shown that perturbative string-independence can be satisfied in lower orders. The conjecture that it can be satisfied at all orders shall is currently under investigation.

# Conclusion

We have obtained two major results. The first one concerns the geometric time ordering of strings in space-time. In order to apply the Epstein-Glaser scheme of renormalization, we have to properly define the time-ordered products of string-localized fields, which play the central role in this approach. That was fully accomplished in chapter 2, where the chopping mechanism was devised to account for the non-intersecting incomparable strings. This result is going to serve as the first step in the study of renormalizability of every quantum field theory perturbatively with string-localized fields. This general setting was posponed for a future publication and is currently being contrived.

The second major result concerns the construction of time-ordered products of linear string-localized fields, which will serve for the general construction with arbitrary Wick monomials of fields. It was shown, as an example in the last section, that even for the simplest non-linear case, the time-ordering product of Wick monomials would lead to an ill-defined time-ordering due to its poly-localization on a set of strings and points. However, we conjecture that the requirement of the string independence condition as a further normalization condition fixes the time-ordering outside the large string diagonal $\Delta_n$.

In addition to the study of time-ordered products of string-localized fields, we have treated in a succint, yet sufficiently compreensible way, the study of renormalizability of point-localized quantum field theories with the example of a scalar field theory with interaction lagrangean $L(x) =: \varphi(x)^n :$. Furthermore, it should be pointed out that the concept of string-localized fields has only recently been rigorously established and the area is very fertile. Several applications such as the study of anomalies and the description of dark matter are already being considered.

# Appendix

# APPENDIX A – Basic Geometric Notions

**Definition A.0.1.** *Given a set $A \subset \mathcal{M}$, we say that $A^c$ is the causal complement of $A$ if*

$$A^c := \{x \in \mathcal{M} : (x-y)^2 < 0, \forall y \in A\}. \tag{A.1}$$

**Definition A.0.2.** *Let $x, y \in \mathcal{M}$ be such that $y \in V_+(x)$. Then, we define the open double cone $D(y,x)$ with $x$ and $y$ as apices by*

$$D(y,x) = V_+(x) \cap V_-(y). \tag{A.2}$$

**Definition A.0.3.** *A hyperplane is a three-dimensional linear submanifold $\Sigma$ of $\mathbb{R}^4$. A hyperplane $\Sigma$ is determined by its normal $u$ (which is a non-zero vector in $\mathcal{M}$) and any one of its points $a$ by*

$$\Sigma := a + u^{\perp} = \{x \in \mathcal{M} : (x-a) \cdot u = 0\}. \tag{A.3}$$

*We say $\Sigma$ is **space-like** if $u$ is time-like, that $\Sigma$ is **time-like** if $u$ is space-like, and that $\Sigma$ is **light-like** if $u$ is light-like.*

Let $e, e'$ be space-like unit vectors, *i.e.* $e \cdot e = -1$.

**Lemma A.0.4.** *i) The linear span of $e, e'$ is time-like / light-like / space-like, respectively, if and only of*

$$(e \cdot e')^2 - 1$$

*is positive / zero / negative, respectively.*

*ii) It is time-like if and only if one of the vectors $e \pm e'$ is time-like and the other one is space-like; It is light-like if and only if one of the vectors $e \pm e'$ is light-like; It is space-like if and only if the vectors $e \pm e'$ are either both time-like or both space-like.*

*Proof.* The first statement is readily verified. Now note that

$$(e \mp e')^2 = -2(1 \pm e \cdot e'),$$

hence

$$(e \cdot e')^2 - 1 \equiv (e \cdot e' + 1)(e \cdot e' - 1) = \frac{-1}{4}(e - e')^2\,(e + e')^2.$$

Thus, $(i)$ implies $(ii)$. □

# APPENDIX B – Extension of Distributions and Scaling Degree

## B.1 Basic Notions on the Extension of Distributions and Scaling Degree

As mentioned in section 3.1 the problem of ultraviolet divergence can be solved by the extension of certain distributions which in turn will account for the renormalization of the theory. In order to do that one uses the notion of **scaling degree** of a distribution. In this section[1], we shall introduce some properties of the scaling degree and provide some simple examples. Let us following definition:

**Definition B.1.1.** *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$, then we define a* ***dilatation*** *$\Lambda$ of the function $\varphi$ through $\lambda$ by*

$$\begin{aligned} \Lambda : \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) &\longrightarrow \mathcal{D}(\mathbb{R}^n) \\ (\lambda, \varphi) &\longmapsto \varphi^\lambda \doteq \lambda^{-n}\varphi(\lambda^{-1}\cdot) \end{aligned} \tag{B.1}$$

*By pullback[2], we can define a* ***dilatation*** *of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ as*

$$(\Lambda^* u)(\varphi) \doteq u_\lambda(\varphi) \doteq u(\varphi^\lambda). \tag{B.2}$$

For the case of $u \in L^1_{loc}(\mathbb{R}^n)$ we can write equation (B.2) as the integral

$$u_\lambda(\varphi) = \int u(\lambda x)\varphi(x) d^n x, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{B.3}$$

The quantity $u(x)$ is referred to as the integral kernel of $u$ and we shall, by the usual abuse of notation, denote a general distribution $u(\varphi)$ as the integral in (B.3).
Let $\mathcal{D}(\mathbb{R}^n \setminus \{0\}) = \{\varphi \in \mathcal{D}(\mathbb{R}^n) : 0 \notin supp(\varphi)\}$ be the subspace of test functions whose support does not contain the origin and $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ its dual[3]. With all that said, the central problem can be stated as follows:

**Problem B.1.2.** *Given a distribution $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, how can we construct a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $u_0(\varphi) = u(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$?*

[1] For more detailed description of the problem of extension of distributions consult [14, 26, 27].

[2] The pullback essencialy transfers the effect of an operation over the test functions to an operation over the distributions by a composition. That is, let $X$ and $Y$ be open sets, $\Lambda$ be the mapping $\Lambda : \mathbb{R}_+ \times X \subseteq \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) \to Y \subseteq \mathcal{D}(\mathbb{R}^n)$ and $u$ be the distribution $u : Y \subseteq \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$, then $u \circ \Lambda := \Lambda^* u : \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$.

[3] The extension exists by Hahn-Banach theorem, see [23, Chap. 3.2].

To answer that question it is necessary to use the concept of the so called scaling degree of a distribution, which basicly measures how singular a distribution is at the origin.

**Definition B.1.3.** *A distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ has* ***scaling degree*** *$s$ (in symbols $sd(u) = s$) with respect to the origin in $\mathbb{R}^n$ if*

$$s \doteq \inf\{s' \in \mathbb{R} : \lambda^{s'} u_\lambda \xrightarrow{\lambda \to 0} 0\}. \tag{B.4}$$

Let us see some simple examples.

**Example B.1.4.** *Let $f \in C^0(\mathbb{R})$ with $f(0) \neq 0$, then*

$$\lim_{\lambda \downarrow 0} \lambda^\omega \int f(\lambda x)\varphi(x)dx = \begin{cases} 0, & if\ \omega > 0 \\ f(0)\int \varphi(x)dx, & if\ \omega = 0 \\ \infty, & if\ \omega > 0 \end{cases} \tag{B.5}$$

*and consequently, $sd(f) = 0$.*

**Example B.1.5.** *Let $\delta \in \mathcal{D}'(\mathbb{R}^n)$ be the Dirac $\delta$-function. Since $\delta(\lambda x) = \lambda^{-n}\delta(x)$, we have $sd(\delta) = n$.*

**Example B.1.6.** *The functions $f(x) = e^{-\frac{1}{x^2}}$ and $g(x) = e^{\frac{1}{x}}$ both define distributions on $\mathbb{R} \setminus \{0\}$. Thus,*

$$\begin{cases} \lim_{\lambda \downarrow 0} \lambda^s f(\lambda x) = 0 \quad \forall s \in \mathbb{R} \Longleftrightarrow sd(f) = -\infty \\ \lim_{\lambda \downarrow 0} \lambda^s g(\lambda x) = \infty \quad \forall s \in \mathbb{R} \Longleftrightarrow sd(g) = \infty \end{cases} \tag{B.6}$$

**Example B.1.7.** *Consider the scalar Feynman propagator $\Delta_F$ in four dimensions given by*

$$\Delta_F(x, m) = \lim_{\varepsilon \downarrow 0} (2\pi)^{-4} \int d^4p \frac{e^{-ip\cdot x}}{(p^2 - m^2 + i\varepsilon)}, \tag{B.7}$$

*then*

$$\begin{aligned} \Delta_F(\lambda x, m) &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-4} \int d^4p \frac{e^{-ip\cdot\lambda x}}{(p^2 - m^2 + i\varepsilon)} \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-4}\lambda^{-2} \int d^4p \frac{e^{-ip\cdot x}}{(p^2 - (\lambda m)^2 + i\varepsilon)} \\ &= \lambda^{-2}\Delta_F(x, \lambda m). \end{aligned} \tag{B.8}$$

*Since for $m \to 0$, $\Delta_F(x, m)$ converges to the massless scalar propagator of the theory, we have $sd(\Delta_F) = 2$.*

The following theorem highlights some important properties of the scaling degree. It will be presented without proof (see [14]).

**Theorem B.1.8.** *Let $t \in \mathcal{D}'(\mathbb{R}^n)$ have $sd(t) = s$ at $0$, then the scaling degree obeys the following properties:*

*i* $sd(\partial^\alpha t) \leq s + |\alpha|$, *where $\alpha \in \mathbb{N}^n$ is any multiindex,*

*ii* $sd(x^\alpha t) \leq s - |\alpha|$, *where $\alpha \in \mathbb{N}^n$ is any multiindex,*

*iii* $sd(ft) \leq sd(t)$, *where $f \in \mathcal{E}^n(\mathbb{R}^n)$,*

*iv* $sd(t_1 \otimes t_2) = sd(t_1) + sd(t_2)$, *for $t_i \in \mathcal{D}'(\mathbb{R}^{n_i})$, $i = 1, 2$.*

A precise answer to problem B.1 is given by the following theorem, which will be presented without proof[4].

**Theorem B.1.9.** *Let $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$*[5].

1. *If $sd(u_0) < n$, then there exists a unique extension $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $sd(u_0) = sd(u)$.*
2. *If $n \leq sd(u_0) < \infty$, then there exist several extensions $u \in \mathcal{D}'(\mathbb{R}^n)$ such that $sd(u_0) = sd(u)$. Given a particular solution $u_p$, the most general solution reads*
$$u = u_p + \sum_{|\alpha| \leq sd(u_0) - n} c_\alpha \partial^\alpha \delta^{(n)} \tag{B.9}$$
*with arbitray constants $c_\alpha \in \mathbb{C}$.*
3. *If $sd(u_0) = \infty$, then there exists no extension $u \in \mathcal{D}'(\mathbb{R}^n)$.*

It is convinient to define the notion of **degree of divergence** of a distribution $u$, which is given by $\mathrm{div}(u) \doteq \mathrm{sd}(u) - n$. It is worth mentioning that the non-unique extension case is given by (B.9) due to the fact that the most general distribution supported at the origin is given by an arbitrary differential polynomial applied to the $\delta$ distribution, see example 1.2.19.

## B.2 Extension of a String-localized Feynman Propagator across the String Diagonal

In Prop. 3.2.1, we have seen that the time-ordered product $\mathrm{T}\,\varphi\varphi$ is fixed outside the string diagonal $\Delta_2$. We illustrate here the extension across $\Delta_2$ with a concrete example, which motivates that one should take the string-localized fields as basic building blocks even though they have been introduced as integrals (6) over point-localized fields.

4 For a detailed proof see [14].

5 In the case the extension is not through the origin, but through an arbitrary submanifold $D$ of $\mathbb{R}^n$ we must replace $n$ in this theorem by the codimension of the submanifold, which is defined as $\mathrm{codim}(D) = \dim(\mathbb{R}^n) - \dim(D) = n - \dim(D)$.

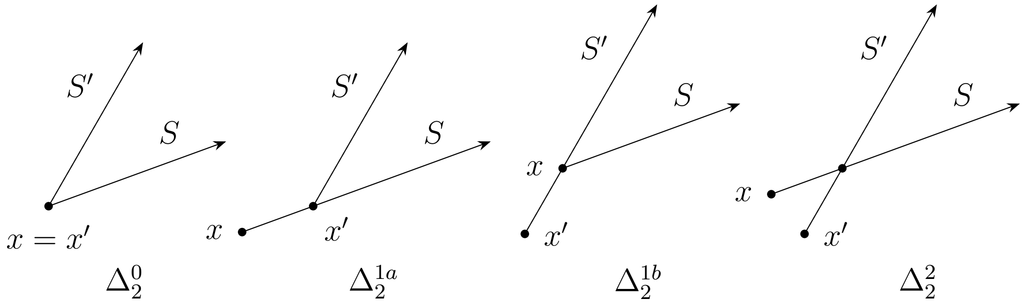


**Figure 6** – Configurations in submanifolds of the string diagonal $\Delta_2$

The string diagonal $\Delta_2$ decomposes into the following disjoint submanifolds:

$$\begin{aligned}
\Delta_2^0 &= \{\, (x, e, x', e') : x = x' \,\}\,, \\
\Delta_2^{1a} &= \{\, (x, e, x', e') : \exists r > 0 \text{ with } x' = x + re \,\}, \\
\Delta_2^{1b} &= \{\, (x, e, x', e') : \exists r' > 0 \text{ with } x = x' + r'e' \,\}, \\
\Delta_2^2 &= \{\, (x, e, x', e') : e, e' \text{ lin. indep.} \wedge \exists r, r' > 0 \text{ with } x + re = x' + r'e' \,\}.
\end{aligned}$$

Here $\Delta_2^0$ consists of the pairs of strings with the same initial point (the point-diagonal); $\Delta_2^{1a}$ is the set of configurations where $x'$ lies in the relative interior of the string $S_{x,e}$, i.e., $x' \in S_{x,e} \setminus \{x\}$; and $\Delta_2^2$ is the set of pairs of strings whose interiors cross: see Fig. 6. Thus $\Delta_2^0$ is the boundary of either $\Delta_2^{1a}$ or $\Delta_2^{1b}$, and $\Delta_2^1 \doteq \Delta_2^{1a} \cup \Delta_2^{1b}$ is the boundary of $\Delta_2^2$. So one must extend the Feynman propagator successively across $\Delta_2^2$; then $\Delta_2^{1a}$ and $\Delta_2^{1b}$; and finally across $\Delta_2^0$.

As an example we consider massive particles of spin one and take a line integral over the Proca field $A_\mu^{\mathrm{p}}$, the so-called escort field [4, 5]:

$$\phi(x, e) \doteq \int_0^\infty ds\, A_\mu^{\mathrm{p}}(x + se)\, e^\mu. \tag{B.10}$$

Its two-point function $\langle \mathrm{T}\, \phi(x, e)\phi(x', e') \rangle$ in momentum space [4] is

$$\frac{1}{m^2} - \frac{e \cdot e'}{(p \cdot e - i\varepsilon)(p \cdot e' + i\varepsilon)}$$

times the on-shell delta distribution $\delta(p^2 - m^2)\theta(p_0)$. It has scaling degree 0 with respect to $\Delta_2^2$ and $\Delta_2^1$, and scaling degree 2 with respect to $\Delta_2^2$ due to the first term. The same holds for the Feynman propagator (outside $\Delta_2$), and its extension across $\Delta_2$ may not exceed this (that is the basic renormalization condition). For all three submanifolds, the codimension is larger than the respective scaling degree, namely 2, 3 and 4, respectively. Therefore the respective extensions are unique [15], and the Feynman propagator is fixed without any freedom: it is defined by replacing $\delta(p^2 - m^2)\theta(p_0)$ by $i/[2\pi(p^2 - m^2 + i\varepsilon)^{-1}]$.

On the other hand, the two-point function of the Proca field in momentum space is $\left(-g_{\mu\nu} + p_\mu p_\nu/m^2\right)$ times the on-shell delta distribution. Its scaling degree with respect to the origin is 4; hence the Feynman propagator admits a renormalization of the form

$$c\, g_{\mu\nu}\, \delta(x - x') \tag{B.11}$$

as is well known. So, if one defines the Feynman propagator as the integral

$$\int_0^\infty ds \int_0^\infty ds'\, \langle \mathrm{T}\, A^{\mathrm{p}}_\mu(x + se) A^{\mathrm{p}}_\nu(x' + s'e') \rangle\, e^\mu e'^\nu, \tag{B.12}$$

as one might be tempted to do from Eq. (B.10), then by (B.11) one has the freedom of adding the distribution

$$c\, e \cdot e' \int_0^\infty ds \int_0^\infty ds'\, \delta(x + se - x' - s'e),$$

supported on $\Delta_2^2$: One has an undetermined constant and therefore has gained nothing, in contrast to the first approach where one takes $\phi(x, e)$ as basic building block.

## B.3 Free Fields for Massive Vector Bosons

In this section we simply give a list of all two-point functions that are relevant to this work. For more detailed discussions of the two-point functions see [4].

Let $\varphi_1, \varphi_2 \in \{A^{\mathrm{p}}_\mu, A_\nu, \phi\}$, then the two-point functions are of the form

$$(\,\Omega, \varphi_1(x, e)\varphi_2(x', e')\Omega\,) = (2\pi)^{-3} \int_{H_m^+} d\mu(p)\; e^{-ip\cdot(x - x')}\, M^{\varphi_1\varphi_2}(p, e, e')\,,$$

where

$$M^{AA}_{\mu,\mu'}(p, e, e') = -g_{\mu\mu'} - \frac{p_\mu p_{\mu'}\,(e \cdot e')}{(p \cdot e - i\varepsilon)(p \cdot e' + i\varepsilon)} + \frac{p_\mu e_{\mu'}}{p \cdot e - i\varepsilon} + \frac{p_{\mu'} e'_\mu}{p \cdot e' + i\varepsilon} \tag{B.13}$$

$$M^{AA^{\mathrm{p}}}_{\mu,\mu'}(p, e) = -g_{\mu\mu'} + \frac{p_\mu e_{\mu'}}{p \cdot e - i\varepsilon} \tag{B.14}$$

$$M^{A\phi}_\mu(p, e, e') = \frac{1}{i}\Big(\frac{e'_\mu}{p \cdot e' + i\varepsilon} - \frac{p_\mu\, e \cdot e'}{(p \cdot e - i\varepsilon)(p \cdot e' + i\varepsilon)}\Big) \tag{B.15}$$

$$M^{A^{\mathrm{p}}A^{\mathrm{p}}}_{\mu,\mu'}(p) = -g_{\mu\mu'} + \frac{p_\mu p_{\mu'}}{m^2} \tag{B.16}$$

$$M^{A^{\mathrm{p}}\phi}_\mu(p, e') = i\Big(\frac{p_\mu}{m^2} - \frac{e'_\mu}{p \cdot e' + i\varepsilon}\Big) \tag{B.17}$$

$$M^{\phi\phi}(p, e, e') = \frac{1}{m^2} - \frac{e \cdot e'}{(p \cdot e - i\varepsilon)(p \cdot e' + i\varepsilon)}. \tag{B.18}$$

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