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“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.” **Arthur Conan Doyle**

Resumo

As exigências de localidade, positividade dos estados e positividade da energia dão origem a comportamentos ruins dos campos quânticos em distâncias pequenas (singularidades UV). Quando tenta-se construir campos quânticos para partículas de spin $s \geq 1$ que satisfazem esse princípios fundamentais, acaba-se ganhando interações não-renormalizaveis. Para spins um e dois, existem campos, no contexto de teorias de calibre, com o mesmo bom comportamento UV que o campo escalar para spin zero. Entretanto, é necessária a introdução de um espaço de estados não-físico, assim como campos não-físicos (ghosts). Motivado por trabalhos anteriores, nós investigamos campos quânticos, para bósons massivos de spin arbitrário, possuindo o mesmo comportamento UV que o campo escalar ($s = 0$), porém que agem num espaço de Hilbert sem ghosts e são covariantes por transformações de Poincaré. Esses campos, entretanto, não possuem mais localização pontual, estando localizados, ao invés, em semi-retas no espaço de Minkowski que se extendem em direções tipo-luz (strings tipo-luz).
Palavras-chave: Campos quânticos. Localização em strings. Localização em strings tipo-luz. Singularidades das funções de dois pontos.

Abstract

The combined requirements of locality, positivity of states and positivity of energy lead to bad short distance behaviour of quantum fields (UV singularities). When one tries to build quantum fields for particles of spin $s \geq 1$ that still satisfy these fundamental principles, one ends up with non-renormalizable interactions. For spin one and two, there exist fields in the context of gauge theory with the same good UV behaviour as the scalar field for spin zero. However, for this one has to introduce an unphysical state space, as well as unphysical fields (ghosts). Motivated by previous works, we begin to investigate quantum fields, for massive bosons of any spin, that have the same good UV behaviour as the scalar field ($s = 0$), act in a Hilbert space without ghosts and are Poincaré covariant. These fields are, however, no longer point-local, being localized instead on semi-infinite lines in Minkowski space extending to lightlike infinity (lightlike strings).
Keywords:Quantum Fields. String-localization. Lightlike string-localization. Two-point functions singularities

Contents

Introduction

One can say that quantum field theory tries to combine the principles of quantum physics and classical relativistic field theory, e.g. eletromagnetism. In this point of view, fields are the necessary objects to implement the interaction between matter particles. An interesting result that hints at this is the so-called ”No-interaction” theorem [1]. The main realm of quantum field theory, is the theory of elementary particles, that speaks of the fundamental components of matter and its interactions. Furthermore, this theory often presents an incredible agreement between theoy and experiment. This success comes from a broad arsenal of powerful modelling methods, based on quantization of classical interactions and the gauge principle. However, these methods often tend to break the fundamental principles inherited from relativity and quantum physics.

From quantum physics, one has that the states of a system (electron in a Hydrogen atom, electron in a metal, photons...) must be described by vectors in some Hilbert space and the observable quantities are represented by self-adjoint operators on this space. Relativity talks about Spacetime, its causality structure and how different observers access the infomation on a system. For short, one has basically three fundamental principles

- Relativistic Covariance: tells how different observers extract infomation from a system.
- Locality (Einstein causality)
- Positivity of the states (Hilbert space positivity): for a probabilistic interpretation of measurable quantities

However, these principles lead to bad short distance behaviour of the quantum fields (ultraviolet singularities), which destroys the predictivity of models describing the interaction of some types of particles. Well-known prescriptions for getting quantum fields with better UV (ultraviolet) behaviour usually breaks the principles, e.g. introduction of unphysical state space (non-positivity) and unphysical fields [2]. One alternative that keeps the Hilbert space positivity and covariance, yielding a good UV behaviour are the so-called spacelike String-local fields [3, 4]. In this work we begin to explore the alternative, hinted in [4, footnote 3], of lightlike string-localized fields, for massive bosons.

This dissertation is divided in two chapters. The first chapter starts with a section revising key concepts on the structure of spacetime, how relativistic covariance enters in quantum mechanics and the description of multi-particle systems. Then, we talk about the usual point-local fields and use them to ilustrate the fundamental principles and important aspects related to interaction, such as renormalizability. Following that we introduce the lightlike String-local fields.

The second chapter is divided into three sections. The first section contains the geometrical (Minkowski geometry) results on lightlike strings such as the time ordering, causal separation, separation by wegdes, among others. In the second section we construct the lightlike string-localized fields from the concept of Wigner intertwiners and explore the particular case of string fields given by line integral over pointlike fields. In the third section is the result on the scaling degree the two-point functions of the lightlike string fields.

1 General Aspects of Quantum Fields

1.1 Preliminaries

In this section we give a brief overview of key aspects of the theory of relativity and how it steps in quantum physics by giving a characterization of fundamental particles [5]. We also introduce the structure of Fock Space, which is needed for studying systems with a undetermined number of particles.

1.1.1 Relativistic spacetime

When working with quantum field theory (QFT) in the contex of high energy physics it is essential that we take into account the relativistic nature of particles (in contrast, many applications of QFT in condensed matter physics do not consider this relativistic nature). Hence we start by giving a overview of the key concepts of the theory of special relativity.

The theory of relativity abandons the galilean view of independent space and time, and consider them as parts of a new entity, namely spacetime. Points in spacetime, called events, are idealizations of real world (objective) occurences happening in a small region of space and taking an instant. Denoting spacetime by $\mathbb{M}$, an event $x \in \mathbb{M}$ can be described by an element of $\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$,

$$x = (x^0, x^1, x^2, x^3) \tag{1.1}$$

with $x^0 = ct$ and $\vec{x} = (x^1, x^2, x^3)$, where c is the speed of light, t is the time and $\vec{x}$ is the position vector relative to a frame of reference. The fundamental principle of special relativity is the constancy of the speed of light with respect to all inertial reference frames (inertial observers), ie if

$$(x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2 = 0^1 \tag{1.2}$$

in one frame of reference then this also holds in another uniformly moving frame

$$(x'^0 - y'^0)^2 - |\vec{x}' - \vec{y}'|^2 = 0 \tag{1.3}$$

Using this, one can prove that the quadratic form q (spacetime interval), given in one frame by

$$q(x, y) = (x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2, \text{ where } x, y \in \mathbb{M} \tag{1.4}$$

[1] The events x,y satisfiying this condition are those who can be connected by a light ray

does not depend on the choice of the frame, ie

$$(x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2 = q(x) \stackrel{!}{=} q'(x) = (x'^0 - y'^0)^2 - |\vec{x}' - \vec{y}'|^2 \tag{1.5}$$

Apart from events, one can also define vectors in spacetime (also called four-vectors) in the following way:
First, given two pairs of events $(x_1, y_1), (x_2, y_2) \in \mathbb{M}^2$, define equilavence relation

$$(x_1, y_1) \sim (x_2, y_2) :\Leftrightarrow y_1^\mu - x_1^\mu = y_2^\mu - x_2^\mu, \text{ for } \mu = 0, 1, 2, 3 \tag{1.6}$$

here μ labels coordinates for the events with respect to some reference frame. This relation is well-defined if one only allows linear inhomogeneous change of coordinates [2] (further on will be clear that this restriction is sufficient). Second, define a four-vector as a equilavence class of this relation and denote the space of all four-vector by $\mathbb{D}(\mathbb{M})$.

Now lets look at the transformations that preserve this quadratic form (that relate different uniformly moving frames),ie the transformations L such that

$$\forall x \in \mathbb{M} : q(Lx, Ly) = q(x, y) \tag{1.8}$$

The set of all such transformations form the so-called Poincaré group $\mathcal{P}$. Every Poincaré transformation can be written as

$$(Lx)^\mu = a^\mu + \Lambda^\mu_\nu x^\nu \tag{1.9}$$

where $a \in \mathbb{R}^4$ and Λ is a (linear) Poincaré transformation that leaves the origin fixed, called a Lorentz transformation.
Given two Poincaré transformations (a_1, Λ_1) and (a_2, Λ_2), their group product is

$$(a_1, \Lambda_1)(a_1, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \tag{1.10}$$

Let $x, y \in \mathbb{M}$ and denote the four-vector they form by $\overrightarrow{(x, y)}$. Notice that every Poincaré transformation is a linear inhomogeneous transformation, hence our definition of four-vectors holds in every inertial frame, furthermore $q(x, y)$ only depends on the equilavence class (vector), allowing one to define $q\left(\overrightarrow{(x, y)}\right) := q(x, y)$. One important remark is that given a reference frame, one can identify $\mathbb{M}$ to $\mathbb{D}(\mathbb{M})$ (and both to $\mathbb{R}^4$) [3]

The quadratic form gives rise to a indefinite inner product. Given $v, w \in \mathbb{D}(\mathbb{M})$,

$$v \cdot w := \frac{1}{4} \{q(v + w) - q(v - w)\}, \text{ where } x, y \in \mathbb{M} \tag{1.11}$$

[2] A linear inhomogeneous transformation of coordinates is of the form:

$$x'^\nu = A^\nu_\mu x^\mu + b^\nu \tag{1.7}$$

where A is a invertible matrix and $b \in \mathbb{R}^4$.

[3] Let $x, y \in \mathbb{M}$. Given a reference frame, consider the event $o = (0, 0, 0, 0)$ (the origin), then x can be indentified to the four-vector $\overrightarrow{(o, x)}$ and the four-vector $\overrightarrow{(x, y)}$ can be identified to the event with coordinates $(y^0 - x^0, \vec{y} - \vec{x})$ (motivated by this, we will denote the vector $\overrightarrow{(x, y)}$ also by $y - x$).

in a frame of reference it takes the form

$$v \cdot w = v^0 w^0 - \vec{v} \cdot \vec{w} \tag{1.12}$$

and one also has that

$$q(v) = v \cdot v \equiv v^2 \tag{1.13}$$

Spacetime endowed with this invariant quadratic form (and consequently with this inner product) is called Minkowski space.

The Lorentz transformations form a subgroup $\mathcal{L} \subset \mathcal{P}$, the so-called Lorentz group. The transformation $\Lambda \in \mathcal{L}$ can also be characterized by being a linear map that preserves the inner product

$$\forall v, w \in \mathbb{D}(\mathbb{M}) : \Lambda v \cdot \Lambda w = v \cdot w \tag{1.14}$$

The Lorentz group decomposes into the following connected components:

- $\mathcal{L}_+^\uparrow$ (proper orthochronous Lorentz group): $det\Lambda = 1$, $\Lambda_0^0 \geq 1$. These transformations preserve space and time orientation.
- $\mathcal{L}_+^\downarrow$: $det\Lambda = 1$, $\Lambda_0^0 \leq -1$, eg $-\mathbb{1}$
- $\mathcal{L}_-^\uparrow$: $det\Lambda = -1$, $\Lambda_0^0 \geq 1$, eg parity $P(x^0, \vec{x}) = (x^0, -\vec{x})$
- $\mathcal{L}_-^\downarrow$: $det\Lambda = -1$, $\Lambda_0^0 \leq -1$, eg time reflection $T(x^0, \vec{x}) = (-x^0, \vec{x})$

The Poincaré group $\mathcal{P}$ decomposes accordingly, but only elements of the $\mathcal{P}_+^\uparrow$ component correspond to physically realizable transformations, making it the relativistic invariance group.

Let's introduce some key definitions and concepts on the causal structure of spacetime.

Definition 1. *Consider a four-vector $v \in \mathbb{D}(\mathbb{M})$, one says that*

(i) v is timelike if $v^2 \equiv v \cdot v > 0$

(ii) v is lightlike if $v^2 \equiv v \cdot v = 0$

(iii) v is spacelike if $v^2 \equiv v \cdot v < 0$

Definition 2. *Given $x \in \mathbb{M}$ and $R \subset \mathbb{M}$, define*

(i) The future lightcone of x: $V_+(x) := \{y \in \mathbb{M} | (y-x)^2 > 0 \ \wedge \ (y-x)^0 > 0\}$

(ii) The past lightcone of x: $V_-(x) := \{y \in \mathbb{M} | (y-x)^2 > 0 \ \wedge \ (y-x)^0 < 0\}$

(iii) The boundary of the future lightcone of x: $\partial V_+(x) := \{y \in \mathbb{M} | (y-x)^2 = 0 \ \wedge \ (y-x)^0 > 0\}$

(iv) The boundary of the past lightcone of x*:* $\partial V_-(x) := \{y \in \mathbb{M} | (y-x)^2 = 0 \ \wedge \ (y-x)^0 < 0\}$

(v) $\overline{V_\pm(x)} := V_\pm(x) \cup \partial V_\pm(x)$

(vi) The causal complement of R*:* $R' := \{z \in \mathbb{M} | \forall y \in R : (z-y)^2 < 0\}$

Definition 3. *Given* $x, y \in \mathbb{M}$ *and* $R, \tilde{R} \subset \mathbb{M}$*, define the following notions:*

(i) R *and* $\tilde{R}$ *are said to be causally disjoint if* $R \subseteq (\tilde{R})'$ *(*$\Leftrightarrow \tilde{R} \subseteq R'$*)*

(ii) Time ordering of events: y *succeeds* x*, writing* $y \succeq x$*, if* $y \notin \overline{V_-(x)}$

(iii) $R \succeq \tilde{R} :\Leftrightarrow \forall x \in R \ \forall \tilde{x} \in \tilde{R} : x \succeq \tilde{x}$

Remarks:

(i) All these notions are invariant under the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$.

(ii) A $v \in \mathbb{D}(\mathbb{M})$ is said to be causal if $v^2 \geq 0$. One says that a causal four-vector v is future[past]-directed if $v^0 > 0[< 0]$, for some reference frame. Given this characterizations, one can stablish a cone structure in $\mathbb{D}(\mathbb{M})$ [4] defining $V_\pm := \{v \in \mathbb{D}(\mathbb{M}) | v^2 > 0 \ \wedge \ v^0 > 0[< 0]\}$, $\overline{V_\pm} := \{v \in \mathbb{D}(\mathbb{M}) | v^2 \geq 0 \ \wedge \ v^0 > 0[< 0]\}$ and $\partial V_\pm := \{v \in \mathbb{D}(\mathbb{M}) | v^2 = 0 \ \wedge \ v^0 > 0[< 0]\}$

(iii) Consider $v = y - x$ with $x, y \in \mathbb{M}$. If v is timelike, then there exists one observer (velocity smaller than c) that can witness both x and y; if v is lightlike, one has that x and y can be connected by a light signal and; if v is spacelike, then x and y could only be witnessed by a observer traveling faster than light (impossible), in that case, one says that x and y are causally separated (or causally disjoint).

(iv) $y \succeq x$ is equivalent to the existence of a reference frame for which $y^0 > x^0$.

(v) The set $\overline{V_+(x)}$ consists of all events that can be influenced (either by a observer of light signal) by $x \in \mathbb{M}$ and, analogously, $\overline{V_-(x)}$ consists of all events that influence $x \in \mathbb{M}$.
For clarity sake, we stress the difference between the relation $\succeq$ from the relation $\ll$ found in [6, 7]. These authors say that x causally preceeds y ($x \ll y$) if $y \in \overline{V_+(x)}$, and call $\overline{V_+(x)}$ the causal future of x. The adopted definition, $y \succeq x$, contains the case $x \ll y$, but allows also the case where x and y are causally separated.

[4] Actually, in General Relativity the causal structure (cone structure) given by the indefinite inner product is initially defined for four-vectors

1.1.2 Poincaré symmetry in Quantum Mechanics and Integer Spin Representations of the Poincaré Group

Here we provide some key definitions and results on how the theory of relativity is combined with quantum mechanics giving rise to the Wigner characterization of elemetary relativistic quantum systems (particles).

The states of a quantum system are described by rays in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ [5]. A Poincaré transformation $L \in \mathcal{P}_+^\uparrow$ transforms single particle states into single particle states,

$$\hat{\Phi} \mapsto \hat{T}_L\hat{\Phi} = \hat{\Psi} \tag{1.16}$$

The system has Poincaré symmetry (relativistic symmetry) if all physical properties are left invariant under all transformation $\hat{T}_L$, ie for all $\Phi, \Psi \in \mathcal{H}$ the transitions probabilities (ray products)

$$\langle\hat{\Phi}, \hat{\Psi}\rangle := \frac{|\langle\Phi, \Psi\rangle|^2}{\|\Phi\|^2\|\Psi\|^2} \tag{1.17}$$

must remain the same,

$$\langle\hat{T}_L\hat{\Phi}, \hat{T}_L\hat{\Psi}\rangle = \langle\hat{\Phi}, \hat{\Psi}\rangle \tag{1.18}$$

For such transformations $\hat{T}_L$ that leave the ray product invariant, one has the following theorem by Wigner [8].

Theorem 1. *Let $\hat{T}$ be an invertible and ray product preserving map of the rays of a Hilbert space $\mathcal{H}$. Then there is an invertible, $\mathbb{R}-$linear and isometric map $T : \mathcal{H} \to \mathcal{H}$ with the property*

$$\hat{T}\hat{\Phi} = \widehat{T\Phi}, \textit{ for } \Phi \in \mathcal{H} \tag{1.19}$$

T is unique up to a factor of modulus 1 and is either unitary or antiunitary.

For the case of $L \in \mathcal{P}_+^\uparrow$, T_L is unitary. If one performs two Poincaré transformations L_1 and L_2, one has that

$$\hat{T}_{L_1}\hat{T}_{L_2} = \hat{T}_{L_1L_2} \tag{1.20}$$

and for the operators T_L it follows

$$T_{L_1}T_{L_2} = e^{i\omega(L_1,L_2)}T_{L_1L_2}, \text{ with } \omega(L_1, L_2) \in \mathbb{R} \tag{1.21}$$

The map $L \in \mathcal{P}_+^\uparrow \mapsto T_L$ satisfiying the condition above is a so-called ray representation or projective representation of the Poincaré group. Fortunatly, there is another theorem by Wigner and Bargmann[8] that provides a true representation, rather than a projective one.

[5] A ray in a Hilbert space is a set of the form:

$$\hat{\Phi} := \{c\Phi | c \in \mathbb{C}\} \equiv \mathbb{C}\Phi, \text{ where } \Phi \in \mathcal{H} \tag{1.15}$$

Theorem 2. *For every continuous ray representation $L \mapsto \hat{T}_L$ of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ there is a strongly continuous unitary representation U of the twofold covering group $\mathcal{P}^c$ [6] ,such that*

$$\widehat{U(a,A)}\Phi = \hat{T}_{a,\Lambda(A)}\hat{\Phi} \tag{1.29}$$

with the covering map Λ.

Now we give the Wigner definition of a elemetary particle: The Hilbert space of states of a elemetary relativistic particle is the representation space of a irreducible, continuous, unitary positive-energy[7] representation of $\mathcal{P}^c$. What remains is the mathematical problem of determining this representations.

Quoting the result on the classification of the irreducible representation: The unitary irreducible positive-energy representations of $\mathcal{P}_+^\uparrow$ are labeled by the mass $m \geq 0$ and spin $s \in \frac{1}{2}\mathbb{N}_0$. In this dissertation, we will consider only massive particles ($m > 0$) of integer spin ($s \in \mathbb{N}_0$), ie bosons, and thus recall explicitly only these representations.
A integer spin representation $U^{(m,s)}$ can be viewed as a true representation, ie a representation of $\mathcal{P}_+^\uparrow$ and not of the covering group $\mathcal{P}^c$, thus

$$U^{(m,s)} : \mathcal{P}_+^\uparrow \longrightarrow \mathcal{B}(\mathcal{H}^{(m,s)}) \tag{1.30}$$
$$(a,\Lambda) \mapsto U^{(m,s)}(a,\Lambda) \tag{1.31}$$

---

6 The group $\mathcal{P}_+^\uparrow$ is two-fold connected with covering group being the so-called inhomogeneous $SL(2,\mathbb{C})$, which we will denote as $\mathcal{P}^c$. This group consist of pairs (a, A) with a being a translations and $A \in SL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ is the covering group of $\mathcal{L}_+^\uparrow$ with covering homomorphism

$$\Lambda : SL(2,\mathbb{C}) \longrightarrow \mathcal{L}_+^\uparrow \tag{1.22}$$
$$A \mapsto \Lambda(A) \equiv \Lambda_A \tag{1.23}$$

defined in the following way:

1. Consider the bijective linear map

$$x \mapsto \underset{\sim}{x} = x^0\mathbb{1} + \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{1.24}$$

from Minkowski space into the space of hermitian 2x2 matrices, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. It holds that

$$det\underset{\sim}{x} = x \cdot x \equiv x^2 \tag{1.25}$$

2. $\Lambda(A)$ is then defined by:

$$\underset{\sim}{\Lambda_A x} = A\underset{\sim}{x}A^* \tag{1.26}$$

One can see that for $A_1, A_2 \in SL(2,\mathbb{C})$,

$$\Lambda_{A_1A_2} = \Lambda_{A_1}\Lambda_{A_2} \tag{1.27}$$

and

$$\Lambda_{A_1} = \Lambda_{A_2} \Leftrightarrow A_1 = \pm A_2 \tag{1.28}$$

This equation ilustrates the double cover, since A and $-A$ define the same Lorentz transformation.

7 This means that the spectrum of the energy-momentum operators, the generators of translations, is restricted to $\overline{V_+}$

with $\mathcal{H}^{(m,s)}$ being the representation space (ie single-particle space) and $\mathcal{B}(\mathcal{H}^{(m,s)})$ the space of all bounded linear operators on it.
The spin characterizes an irreducible unitary representation $D^{(s)}$ of the little group (stabilizer group) in $\mathcal{L}_+^\uparrow$ of a reference momentum $\bar{p}$ on the mass shell for $m > 0$,

$$H_m^+ := \{p \in \mathbb{R}^4 | p \cdot p = m^2,\, p^0 > 0\} \tag{1.32}$$

This subgroup is the rotation group, $SO(3)$, and the representation $D^{(s)}$ acts in $\mathbb{C}^{2s+1}$, the so-called little Hilbert space. The representation $U^{(m,s)}$ is induced by $D^{(s)}$ as follows:

- $\mathcal{H}^{(m,s)} = L^2\left(H_m^+, d\mu_m; \mathbb{C}^{2s+1}\right)$, where $d\mu_m$ is the Lorentz invariant measure on H_m^+.
- $U^{(m,s)}$ acts according to

$$(U^{(m,s)}(a,\Lambda)\psi)(p) = e^{ip\cdot a} D^{(s)}\left(R(\Lambda,p)\right)\psi(\Lambda^{-1}p) \tag{1.33}$$

  Here $R(\Lambda,p) \in SO(3)$ is the so-called Wigner rotation, defined by

$$R(\Lambda,p) := B_p^{-1}\Lambda B_{\Lambda^{-1}p} \tag{1.34}$$

  where B_p, $p \in H_m^+$, is a family of Lorentz transformations such that $B_p : \bar{p} \mapsto p$.

This representation extends to the full Poincaré group by adjoining representers for the space reflection P (or parity transformation) and the time reflection $T = -P$. Indeed, all integer spin representations $D^{(s)}$ extend to $O(3)$, and by an appropiate choice of the family B_p, the representation $U^{(m,s)}$ extends naturally to the parity transformation by

$$\left(U^{(m,s)}(P)\psi\right)(p) = D^{(s)}(P)\,\psi(Pp). \tag{1.35}$$

Similarly, one can adjoin an anti- unitary representer of the time reflection to the representation of $O(3)$, and one can define an anti-unitary involution $U^{(m,s)}(T)$ by

$$\left(U^{(m,s)}(T)\psi\right)(p) := D^{(s)}(T)\,\psi(-Tp). \tag{1.36}$$

(See [4] for details.) Note that the anti-unitary representer of the PT transformation $PT \equiv -\mathbb{1}$ is now given by

$$\left(U^{(m,s)}(-\mathbb{1})\psi\right)(p) = D^{(s)}(-\mathbb{1})\,\psi(p), \tag{1.37}$$

where $D^{(s)}(-\mathbb{1})$ is the anti-untiary operator $D^{(s)}(T)D^{(s)}(P)$.

Given a reference system, we identify the momentum space to $\mathbb{R}^4$: $p \mapsto (p_0, \cdots, p_3)$. The Lorentz product then reads $p \cdot p = p_0^2 - |\vec{p}|^2$ with $|\vec{p}|^2 = p_1^2 + p_2^2 + p_3^2$. The Lorentz invariant measure on the mass shell is

$$d\mu_m(p) = \frac{d^3\vec{p}}{2\omega_m(\vec{p})}, \quad \omega_m(\vec{p}) \doteq (|\vec{p}|^2 + m^2)^{\frac{1}{2}}. \tag{1.38}$$

We choose the reference system such that the reference momentum $\bar{p}$ is identified with $(m, 0)$ in $\mathbb{R}^4$. Then the space- and time-reflections are given by

$$P : (x^0, x) \mapsto (x^0, -x), \quad T : (x^0, x) \mapsto (-x^0, x). \tag{1.39}$$

1.1.3 Fock space

So far we only talked about single-particle spaces, however is of the most interest to treat systems with an undertermined number of particles, ie systems where one can have creation and annihilation of particles. The method of passing from a single-particle description to a multi-particle description is often called second quantization. We provide some of the key definitions and concepts along the lines of [9, 10].

Consider a arbitrary single-particle Hilbert space, $\mathcal{H}_1$, eg $\mathcal{H}_1 = L^2(\mathbb{R}^3)$, spinless non-relativistic particle, or $\mathcal{H}_1 = \mathcal{H}^{(m,s)}$, spin-$s$ relativistic particle. The Hilbert space that describes a n-particle system is obtained as follows: First, consider the Hilbert space

$$\mathcal{H}_n := \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{\text{n times}} \equiv {\mathcal{H}_1}^{\otimes n} \tag{1.40}$$

A typical vector on $\mathcal{H}_n$ is of the form

$$\Phi_n = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n, \text{ where } \phi_1, \cdots \phi_n \in \mathcal{H}_1 \tag{1.41}$$

or is given by a linear combination of such vectors. The inner product on $\mathcal{H}_n$ is given by

$$(\Psi_n, \Phi_n)_n := (\psi_1, \phi_1)_1 \cdots (\psi_n, \phi_n)_1 \tag{1.42}$$

with $(\cdot,\cdot)_1$ being the inner produc on $\mathcal{H}_1$.
Then, we must take into account that quantum particles are indistinguishable and, consequently, obey either Bose-Einstein or Fermi-Dirac statistics. The states Φ_n must be substituted by either

$$E_n^+ \Phi_n = \frac{1}{n!} \sum_{\pi \in S_n} \phi_{\pi(1)} \otimes \cdots \otimes \phi_{\pi(n)} \tag{1.43}$$

or

$$E_n^- \Phi_n = \frac{1}{n!} \sum_{\pi \in S_n} sgn(\pi) \phi_{\pi(1)} \otimes \cdots \otimes \phi_{\pi(n)} \tag{1.44}$$

where S_n is the symmetric group on n letters and $sgn(\pi)$ is the sign of the permutation $\pi \in S_n$. The operators E_n^+ and E_n^- are projection operators [8] to, respectively, the symmetrized (bosonic) and anti-symmetrized (fermionic) parts of $\mathcal{H}_n$. The true physical n-particle spaces are

$$\mathcal{H}_n^\pm = E_n^\pm {\mathcal{H}_1}^{\otimes n} \tag{1.47}$$

Having the description of multi-particle systems with a fixed number of particles, we now want to describe a system that can be in any multi-particle state (ie a system with

[8] $E_n^\pm$ satisfy:

$$(E_n^\pm)^2 = E_n^\pm \tag{1.45}$$

$$\forall \Phi_n, \Psi_n \in \mathcal{H}_n : (\Psi_n, E_n^\pm \Phi)_n = (E_n^\pm \Psi_n, \Phi)_n \tag{1.46}$$

an undertermined number of particles). In order to do this, one introduces the Fock space

$$\mathcal{H}^{\pm} \equiv \mathcal{F}^{\pm}(\mathcal{H}_1) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{\pm} \tag{1.48}$$

where $\mathcal{H}_0 = \mathbb{C}$. The state $\Omega = (1, 0, 0, \cdots)$ is the so-called vaccum state . An element of $\mathcal{H}^{\pm}$ is an infinite sequence of states

$$\Phi = (c, \Phi_1, \cdots, \Phi_n, \cdots) \tag{1.49}$$

with $c \in \mathbb{C}$ and $\Phi_n \in \mathcal{H}_n^{\pm}$. The Fock space is also a Hilbert space, with inner product

$$(\Phi, \Psi) := \sum_{n=0}^{\infty} (\Psi_n, \Phi_n)_n \tag{1.50}$$

and all $\Phi \in \mathcal{H}^{\pm}$ have finite norm, ie

$$\|\Phi\|^2 = \sum_{n=0}^{\infty} \|\Phi_n\|_n^2 < \infty \tag{1.51}$$

Now we introduce operators some important operators on $\mathcal{H}^{\pm}$. Given a unitary operator U_1 on $\mathcal{H}_1$; eg $U_1 = U^{(m,s)}(a, \Lambda)$, the operator that implements the Poincaré transformation (a, Λ) from last section; one defines the so-called second quantization U of U_1, that acts on $\mathcal{H}^{\pm}$, by

$$U\Omega := \Omega \tag{1.52}$$

$$(U\Phi)_n := (\bigotimes_{j=1}^{n} U_1)\Phi_n \tag{1.53}$$

The operator U does not change the particle number of the system, but there exist operators that do change. The creation operator $a^*(\phi)$ creates a particle in the state $\phi \in \mathcal{H}_1$

$$a^*(\phi)\Omega = (0, \phi, 0, \cdots) \equiv \phi \tag{1.54}$$

$$(a^*(\phi)\Phi)_n = \sqrt{n} E_n^{\pm}(\phi \otimes \Phi_{n-1});\ n = 1, 2, \cdots \tag{1.55}$$

Given the creation operator $a^*(\phi)$, one defines the annihilation operator for a state $\phi \in \mathcal{H}_1$, $a(\phi)$, as its adjoint. The vaccum state is characterized by the property

$$\forall \phi \in \mathcal{H}_1 : a(\phi)\Omega = 0 \tag{1.56}$$

One can also determine the commutation/anticommutation relations for the creation and annihilation operators, those being

$$[a(\phi), a^*(\psi)]_{\mp} = (\phi, \psi)_1 \tag{1.57}$$

$$[a(\phi), a(\psi)]_{\mp} = 0 = [a^*(\phi), a^*(\psi)]_{\mp} \tag{1.58}$$

The $+$ sign stands for commutation (bosons) and the $-$ sign stands for anticommutation (fermions).

As we stated in the last section, in this dissertation we will only consider bosons. An important remark, however, is that we previously refered to bosons as massive particles of integer spin, in contrast to the above definition of them as symmetrized n-particle states. The equilavence of the two characterizations is a general feature of Relativistic Quantum Field Theory, namely the Spin-Statistics theorem [11], which asserts that integer spin particles obey Bose-Einstein statistics and half-integer spin particles obey Fermi-Dirac statistics.

1.2 Free quantum fields

Having determined the state spaces and the relativistic transformation laws of the various types of relativistic particles and constructed their Fock spaces, for the description of multi-particle states, ones aim now is to construct operators that create particle states from the vaccum. These operators are called free quantum fields and are built from creation and annihilation operators. Quantum fields are going to be used for building the S matrix, central object of the description of scattering experiments involving the interactions between the given particles, and for this one requires that the fields satisfy

- Relativistic Covariance
- Locality (or Einstein causality)
- Positivity of the states (for a probabilistic interpretation)

1.2.1 The point-local Scalar and Vector fields

Let's ilustrate these properties with scalar and vector fields, since we are only treating integer spin particles.

(i) Scalar field:

Given the representation $U^{(m,0)}$[9] of a massive spinless particle acting on $L^2(H_m^+, d\mu_m)$ (state space) and writing symbolically the creation and annihilation operators as

$$a^*(\psi) =: \int\limits_{H_m^+} d\mu_m(p)\psi(p)a^*(p) \tag{1.59}$$

$$a(\psi) =: \int\limits_{H_m^+} d\mu_m(p)\overline{\psi(p)}a(p) \tag{1.60}$$

[9] The representation of the little group is trivial

where $\psi \in L^2(H_m^+, d\mu_m)$, one constructs the field as

$$\varphi(x) = (2\pi)^{-\frac{3}{2}} \int\limits_{H_m^+} d\mu_m(p) \left(e^{ip\cdot x} a^*(p) + e^{-ip\cdot x} a(p)\right) \tag{1.61}$$

remembering that this expression must be understood in the sense of distributions. $\varphi(x)$ is an operator-valued distribution in the sense that its matrix elements are tempered distributions [10].

The positivity of states follows from the construction. The scalar field transforms under a trivial representation of $\mathcal{P}_+^\uparrow$

$$U(a,\Lambda)\varphi(x)U(a,\Lambda)^* = \varphi(a+\Lambda x) \tag{1.62}$$

where U is the second quantization of $U^{(m,0)}$. The locality is expressed as follows

$$[\varphi(x), \varphi(y)] = 0, \text{ if } (x-y)^2 < 0 \tag{1.63}$$

, i.e. the field 'operator' commutes with respect to causally disjoint events (that are points in $\mathbb{M}$, thus the denomination point-like fields/localization). An important remark is that the locality of the field is equivalent to the symmetry of the so-called two-point function,

$$w^{scl}(x-y) \equiv (\Omega, \varphi(x)\varphi(y)\Omega) = (2\pi)^{-3} \int\limits_{H_m^+} d\mu_m(p) e^{-ip\cdot(x-y)} \tag{1.64}$$

with respect to the interchange $x \leftrightarrow y$.

(ii) Vector field:

Given the representation $U^{(m,1)}$ of a massive spin-1 particle acting on $L^2(H_m^+, d\mu_m; \mathbb{C}^3)$ (state space), one can construct the so-called Proca field A_μ^p as in [8].
The positivity is also satisfied by construction. The field is hermitean and it transforms under $\mathcal{P}_+$ (covariance) as

$$U(a,\Lambda)A_\mu^p(x)U(a,\Lambda)^* = \Lambda^\nu{}_\mu A_\nu^p(a+\Lambda x), \tag{1.65}$$

A_μ^p is (point-)localized in the same sense as the scalar field

$$[A_\mu^p(x), A_\nu^p(x)] = 0, \text{ if } (x-y)^2 < 0 \tag{1.66}$$

and its two-point function is given by

$$w_{\mu\nu}^p(x-y) \equiv (\Omega, A_\mu^p(x)A_\nu^p(y)\Omega) = (2\pi)^{-3} \int\limits_{H_m^+} d\mu_m(p) e^{-ip\cdot(x-y)} M_{\mu\nu}^p(p) \tag{1.67}$$

where

$$M_{\mu\nu}^p(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \tag{1.68}$$

[10] A tempered distribution is a distribution over the so-called Schwarz space, $\mathcal{S}(\mathbb{R}^4)$, of rapidly decreasing C^∞ functions on $\mathbb{R}^4$

An important remark is that higher spin($s \geq 2$) local covariant free fields can also be contructed using the so-called Wigner intertwiners, which for point-local fields can found in [8, 12].

1.2.2 Considerations on interacting point-local fields and ultraviolet (UV) divergences

The combined requirements of locality, positivity of states and positivity of energy lead to bad short distance behaviour of quantum fields (UV singularities). In the construction of interacting models one starts from a given set of particle types with corresponding free fields and an interaction Lagrangean L_{int}, which is a Wick ordered polynomial of free fields [9] and describes the coupling between the various particles. Let g be a test function. The S matrix is given by the formal series of operators

$$S(gL_{int}) := \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n g(x_1) \cdots g(x_n) T[L_{int}(x_1) \cdots L_{int}(x_n)] \tag{1.69}$$

where $T \cdots$ denotes the time-ordered product, eg

$$T[L_{int}(x)L_{int}(x')] = \begin{cases} L_{int}(x)L_{int}(x') & \text{, se } x \succeq x' \\ L_{int}(x')L_{int}(x) & \text{, se } x' \succeq x \end{cases}$$

The Physical S matrix is obtained by taking the so-called adiabatic limit where $g(x)$ goes to a constant (Infrared problem). The time-ordered distributions are recursively fixed only outside the set of coinciding arguments, eg $T[L_{int}(x)L_{int}(x')]$ is fixed only for $x \neq x'$, and the extension into this set is unique only after specifying some normalization constants (UV problem) , which is done so as to satisfy physically motivated (re-)normalization conditions. If the number of normalization constants increases without bound with the order n of the series, the theory is called non-renormalizable, now if the total number of constants appearing in all orders is finite, then the theory is called renormalizable. A non-renormalizable theory has a weaker predictive power.

These time ordered products $T[L_{int}(x_1) \cdots L_{int}(x_n)]$ can be expanded using Wicks Theorem [9] in a sum involving the time-ordered two-point functions, eg considering $T[L_{int}(x)L_{int}(x')]$ with $L_{int} =: \varphi(x)^3 :$,

$$\begin{aligned} T: \varphi(x)^3 : : \varphi(x')^3 : \; =: \varphi(x)^3 \varphi(x')^3 : \; + 9(\Omega, T\varphi(x)\varphi(x')\Omega) : \varphi(x)^2 \varphi(x')^2 : + \\ + 9(\Omega, T : \varphi(x)^2 : : \varphi(x')^2 : \Omega) : \varphi(x)\varphi(x') : \; + (\Omega, T : \varphi(x)^3 : : \varphi(x')^3 : \Omega) \end{aligned}$$

The UV behaviour of the time-ordered distributions can be traced back to the UV behaviour of the two-point functions, and the UV behaviour of the latter is determined by its so-called scaling degree.

The scaling degree of a two-point function $w(x-x')$ quantifies its singular behaviour when $\xi \equiv x - x' = 0$, ie a scaling degree ω means that w scales like $\lambda^{-\omega}$ under $\xi \mapsto \lambda\xi$. Let us give the precise definition:

Definition 4. *Let $u(\xi)$ be a distribution on $\mathbb{R}^4$. We define the scaling degree of u, denoted by $sd(u)$, as* [11]

$$\underset{\omega\in\mathbb{R}}{inf} \left\{ \forall f \in \mathcal{D}(\mathbb{R}^4) : \lambda^{\omega}\langle u_\lambda, f\rangle \xrightarrow{\lambda\to 0} 0 \right\} \tag{1.70}$$

where the rescaled distribution u_λ is defined as

$$\langle u_\lambda, f\rangle := \langle u, f^\lambda\rangle \textit{ with } f^\lambda(\xi) := \lambda^{-4} f(\lambda^{-1}\xi)$$

The knowledge of the scaling degrees of the two-point functions allow us to calculate the so-called scaling dimension of the interaction Lagrangean. This scaling dimension is defined as follows: the scaling dimension of a free field is half the scaling degree of its two-point function and the scaling dimension of a Wick product of different fields(eg an interaction Lagrangean) is the sum of the scaling dimensions of the fields. The scaling dimension of the interaction Lagrangean allows to inquire about the renormalizability of the given model in the sense that if the lagrangean of a given model (in 4 spacetime dimensions) have scaling dimension larger than 4, then the model is non-renormalizable[9].

The point-like scalar field has a good UV behaviour, since its two-point function scaling degree is 2, however like we stated in the begining of the section, the requirements of locality and positivity (of states and energy) on the fields lead to a bad UV behaviour, more precisely[4, 12], the optimal scaling degree for a spin-s quantum satisfiying the above assumptions is $2s + 2$. The Proca field A^p has scaling dimension 2. The high scaling degrees for these fields excludes its use in the pertubative (S matrix given by a formal series) construction of renormalizable interacting models for $s \geq 1$. For spin one and two, there exist fields in the context of gauge theory with the same good UV behaviour as the scalar field for spin zero (scaling degree 2)[2]. However, for this one has to introduce an unphysical state space (unphysical in the sense that it contains stated with negative norm, not allowing a probabilistic interpretation), as well as unphysical fields (ghosts). In the contruction of interacting models, the unphysical degrees of freedom have to be divided out in the end by requiring gauge (of BRTS) invariance of observables and the S matrix.

1.2.3 The alternative of String-localized Quantum Fields

Like was pointed in the previous section, when one tries to build quantum fields for particles of spin $s \geq 1$ that still satisfy the fundamental conditions of Hilbert space positivity, Poincaré covariance and Einstein causality (locality) one ends up with non-renormalizable interactions. Some ways to avoid the non-renormalizability break one or

[11] $\mathcal{D}(\mathbb{R}^4)$ is the space of compactly supported test functions on $\mathbb{R}^4$

more of the fundamental conditions we stated. However in [4], (continuing the work done on [13, 3]) were constructed free quantum fields for massive bosons of any spin which have the same good UV behaviour of the scalar field, act in a Hilbert space without ghosts and are Poincaré covariant. The price to pay is that these fields are no longer point-local , in the sense of (1.63), but are localized in semi-infinite lines

$$S_{x,e} = x + \mathbb{R}_0^+ e \tag{1.71}$$

where $x \in \mathbb{M}$ and e is a spacelike vector, the so-called (spacelike) Strings. These fields are thus called string-localized fields. In this dissertation we begin to explore the alternative, hinted in [4, footnote 3], of lightlike string-localized fields.

A quantum tensor field localized on lightlike strings is a mutliplett of operator-valued distributions $\varphi_{\mu_1 \cdots \mu_k}(x, e)$, where x is a point in Minkowski space and e is in the forward light cone $\partial V_+ \equiv H_0^+$[12]. It is a distribution in x and a function in e , that is to say, it needs to be smeared only in x. This is an advantage in contrast to spacelike string-local fields, since they must be considered as distributions also in e.

The lightlike string emanating from x in the direction e, $S_{x,e} := x + \mathbb{R}_0^+ e$, is the localization region of $\varphi_{\mu_1 \cdots \mu_k}(x, e)$ in the sense of compatibility of quantum observables: If the strings $S_{x,e}$ and $S_{x'',e'}$ are causally disjoint for all x'' in an open neighborhood of x', then

$$[\varphi_{\mu_1 \cdots \mu_k}(x, e), \varphi_{\mu'_1 \cdots \mu'_k}(x', e')] = 0. \tag{1.72}$$

It is further required that the family transform as a tensor under a unitary representation U of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$:

$$U(a, \Lambda)\, \varphi_{\mu_1 \cdots \mu_k}(x, e)\, U(a, \Lambda)^{-1} = \varphi_{\alpha_1 \cdots \alpha_k}(a + \Lambda x, \Lambda e)\; \Lambda^{\alpha_1}{}_{\mu_1} \cdots \Lambda^{\alpha_k}{}_{\mu_k}, \tag{1.73}$$

where $a \in \mathbb{R}^4$ is a translation and Λ is a Lorentz transformation. (We use Einstein's sum convention in repeated Lorentz indices.) We say that the field is a free field for a given particle type if it creates from the vacuum only single particle states of the given type. Like stated before, we consider here only massive bosons, i.e.particles with positive mass, $m > 0$, and integer spin, $s \in \mathbb{N}_0$.

[12] Note that H_0^+ is the mass-shell for $m = 0$

2 Definitions, constructions and results on Lightlike String-local Quantum Fields

The chapter is divided into three sections. The first section contains the geometrical (Minkowski geometry) results on lightlike strings such as the time ordering, causal separation, separation by wegdes, among others. In the second section we construct the lightlike string-localized fields from the concept of Wigner intertwiners and explore the particular case of string fields given by line integral over pointlike fields. In the third section is the result on the scaling degree of the two-point functions of the lightlike string fields.

2.1 Geometrical results on Strings

We start by remembering the key definitions (2,3 from section 1.1.1) concerning the causal structure on $\mathbb{M}$.
Given $x, y \in \mathbb{M}$ and $R, \tilde{R} \subseteq \mathbb{M}$,

- $y \succeq x :\Leftrightarrow y \notin \overline{V_-(x)}$
- $R \succeq \tilde{R} :\Leftrightarrow \forall x \in R\ \forall \tilde{x} \in \tilde{R} : x \succeq \tilde{x}$
- $R' := \{z \in \mathbb{M} | \forall y \in R : (z-y)^2 < 0\}$
- R and $\tilde{R}$ are causally disjoint if $R \subseteq (\tilde{R})'$

We also introduce the important concept of an wedge following [14, 15]. We give two equivalent characterizations:

- Given a reference frame such that $\mathbb{M} \ni x = (x^0, x^1, x^2, x^3)$, we define the standard Wedge W_1 as
$$W_1 := \left\{x \in \mathbb{R}^4 |\ x^1 > \left|x^0\right|\right\}, \tag{2.1}$$
All other wedges are defined as Poincaré transforms of W_1, ie
$$W = \Lambda W_1 + a \text{ , for } (a, \Lambda) \in \mathcal{P}_+^\uparrow \tag{2.2}$$
- Let $e, e' \in \partial V_+$ linearly independent and $x \in \mathbb{M}$. One defines a wedge as the following set:
$$W(x; e, e') := \{y \in \mathbb{M} | (y-x) \cdot e > 0\ \wedge\ (y-x) \cdot e' < 0\} \tag{2.3}$$

In the following subsections we present our propositions concerning the causal properties of the strings.

2.1.1 Causal complement, causal disjointness and wedge separation

Proposition 1. *Let $S_{x,e}$ be a future-directed lightlike string. Then,*

$$(S_{x,e})' = \Big\{ y \in \mathbb{M} | (y-x)^2 < 0 \ \wedge \ (y-x) \cdot e \geq 0 \Big\}$$

Proof. Let $y \in (S_{x,e})'$, which also reads as $\forall \alpha \in \mathbb{R}_0^+ : [y - (x + \alpha e)]^2 < 0$. However

$$[y - (x + \alpha e)]^2 = (y-x)^2 - 2\alpha(y-x) \cdot e + \alpha^2 \cancelto{0}{e \cdot e} < 0$$

- For $\alpha = 0$, one gets $(y-x)^2 < 0$.
- For $\alpha \neq 0$ one gets $(y-x)^2 - 2\alpha(y-x) \cdot e < 0$, however

  $$(y-x)^2 - 2\alpha(y-x) \cdot e < 0 \Rightarrow (y-x) \cdot e \geq 0$$

  because otherwise there would exist an $\alpha_0 \in \mathbb{R}_0^+$ such that

  $$(y-x)^2 - 2\alpha_0(y-x) \cdot e > 0$$

  which is a contradiction.

Therefore, $(S_{x,e})' \subseteq \{y \in \mathbb{M} | (y-x)^2 < 0 \ \wedge \ (y-x) \cdot e \geq 0\}$.

Now lets prove the other inclusion ($\supseteq$).
Consider $y \in \mathbb{M}$ such that $(y-x)^2 < 0$ and $(y-x) \cdot e \geq 0$. Then,

$$\forall \alpha \in \mathbb{R}_0^+ : [y-(x+\alpha e)]^2 = (y-x)^2 - 2\alpha(y-x) \cdot e + \alpha^2 \cancelto{0}{e \cdot e} = \underbrace{(y-x)^2}_{<0} \underbrace{- \underbrace{2\alpha(y-x) \cdot e}_{\geq 0}}_{\leq 0} < 0$$

Therefore, $(S_{x,e})' \supseteq \{y \in \mathbb{M} | (y-x)^2 < 0 \ \wedge \ (y-x) \cdot e \geq 0\}$ □

Proposition 2. *Let $S_{x,e}$ and $S_{x',e'}$ be future-directed lightlike strings. Then,*

$$S_{x,e} \subset (S_{x',e'})' \Leftrightarrow (x-x')^2 < 0 \ \wedge \ (x'-x) \cdot e \geq 0 \ \wedge \ (x-x').e' \geq 0$$

Proof. Lets prove the two implications:

($\Rightarrow$) Using proposition 1, one gets:

$$S_{x,e} \subset (S_{x',e'})' \Leftrightarrow \forall \alpha \in \mathbb{R}_0^+ \text{:} \ \underbrace{[(x+\alpha e) - x']^2 < 0}_{(i)} \ \wedge \ \underbrace{[(x+\alpha e) - x'] \cdot e' \geq 0}_{(ii)}$$

(i) $\forall \alpha \in \mathbb{R}_0^+$: $(x-x')^2 + 2\alpha(x-x')\cdot e + \alpha^2 \cancelto{0}{e\cdot e} < 0$

- For all $\alpha = 0$, one gets $(x-x')^2 < 0$
- For all $\alpha \neq 0$, one gets $(x-x').e \leq 0$ ($\Leftrightarrow (x'-x).e \geq 0$)
  (because otherwise there would exist a $\alpha_0 \in \mathbb{R}_0^+$ s.t. $(i) \geq 0$.)

(ii) $\forall \alpha \in \mathbb{R}_0^+$: $(x-x')\cdot e' + \alpha \overbrace{e\cdot e'}^{>0} \geq 0$

- For $\alpha = 0$, one gets $(x-x')\cdot e' \geq 0$

($\Leftarrow$) Let $x, x' \in \mathbb{M}$ and $e, e' \in \mathbb{D}(\mathbb{M})$ such that

$$(x-x')^2 < 0 \ \wedge \ (x'-x)\cdot e \geq 0 \ \wedge \ (x-x').e' \geq 0$$

and take $\alpha \in \mathbb{R}_0^+$. One then gets

- $[(x+\alpha e) - x']^2 = (x-x')^2 + 2\alpha(x-x')\cdot e = (x-x')^2 + \underbrace{2\alpha}_{>0}\underbrace{(x-x')\cdot e}_{\leq 0} < 0$
- $[(x+\alpha e) - x']\cdot e' = \underbrace{(x-x')\cdot e'}_{\geq 0} + \alpha \underbrace{e\cdot e'}_{>0} \geq 0$

Therefore, $S_{x,e} \subset (S_{x',e'})'$ $\square$

Corollary 1. *Let $S_{x,e}$ and $S_{x',e'}$ be future-directed lightlike strings. There exists a neighborhood U' of x' such that*

$$\forall x'' \in U' : S_{x,e} \subset (S_{x'',e'})',$$

if and only if,

$$(x-x')^2 < 0 \ \wedge \ (x'-x)\cdot e > 0 \ \wedge \ (x-x').e' > 0$$

Proof. ($\Leftarrow$) Consider the following functions $q_i : x'' \in \mathbb{M} \mapsto q_i(x'') \in \mathbb{R}$, $i = 1,2,3$, given by

$$\begin{cases} q_1(x'') = (x''-x)^2 \\ q_2(x'') = (x''-x)\cdot e \\ q_3(x'') = (x-x'')\cdot e' \end{cases}$$

One has that q_1, q_2, q_3 are all continuous functions and furthermore, $q_1(x') < 0$ and $q_{2,3} > 0$. Since continuous functions preserve its sign in a neighborhood of a point, there exists a neighborhood U' of x' such that

$$\forall x'' \in U' : q_1(x'') < 0 \ \wedge \ q_2(x'') > 0 \ \wedge \ q_3(x'') > 0$$

which implies that

$$\forall x'' \in U' : S_{x,e} \subset (S_{x'',e'})'$$

($\Rightarrow$) One has that

$$\forall x'' \in U' : \underbrace{(x-x'')^2 < 0}(I) \;\wedge\; \underbrace{(x''-x)\cdot e \geq 0}(II) \;\wedge\; \underbrace{(x-x'').e' \geq 0}(III)$$

The first condition, (I), gives $(x-x')^2 < 0$, since $x' \in U'$. By the same token, conditions (II) and (III) yield

$$(x'-x)\cdot e \geq 0 \;\wedge\; (x-x').e' \geq 0$$

Note that one cannot have $(x'-x)\cdot e = 0 = (x-x').e'$, since, in this case, the open neighborhood U' of x' would contain points x'' satisfiying $(x-x'').e' < 0$ and/or $(x''-x).e < 0$, which is a contradiction. □

Proposition 3. *Let $S_{x,e}$ and $S_{x',e'}$ be non-parallel, causally disjoint, future-directed lightlike strings. Then, there exists a pair of wegdes W_L, W_R such that $W_L' = \overline{W_R}$, $W_R' = \overline{W_L}$, $S_{x,e} \subseteq \overline{W}_L$ and $S_{x',e'} \subseteq \overline{W}_R$.*

Proof. By proposition 2, the fact that the strings are causally disjoint reads as

$$S_{x,e} \subset (S_{x',e'})' \Leftrightarrow (x-x')^2 < 0 \;\wedge\; (x'-x)\cdot e \geq 0 \;\wedge\; (x-x').e' \geq 0$$

Furthermore, one must have $e \neq e'$, since the lightlike strings are non-parallel. Lets explicitly contruct the pair of wedges. Define

$$W_R := \{y \in \mathbb{M} | [y - (\frac{x+x'}{2})]\cdot e > 0 \;\wedge\; [y - (\frac{x+x'}{2})]\cdot e' < 0\}$$
$$W_L := \{y \in \mathbb{M} | [y - (\frac{x+x'}{2})]\cdot e < 0 \;\wedge\; [y - (\frac{x+x'}{2})]\cdot e' > 0\}$$

Notice that $W_L' = \overline{W_R}$, $W_R' = \overline{W_L}$ and that:

- $S_{x',e'} \subseteq \overline{W}_R$

  Let $\alpha \in \mathbb{R}_0^+$.

$$[(x'+\alpha e') - (\frac{x+x'}{2})]\cdot e = [\frac{x'-x}{2} + \alpha e']\cdot e = \frac{1}{2}\overbrace{(x'-x)\cdot e}^{\geq 0} + \overbrace{\alpha e'\cdot e}^{\geq 0} \geq 0$$
$$[(x'+\alpha e') - (\frac{x+x'}{2})]\cdot e' = [\frac{x'-x}{2} + \alpha e']\cdot e' = \frac{1}{2}\overbrace{(x'-x)\cdot e'}^{\leq 0} + \alpha\overbrace{e'\cdot e}^{=0} \leq 0$$

- $S_{x,e} \subseteq \overline{W}_L$

  Let $\alpha \in \mathbb{R}_0^+$.

$$[(x+\alpha e) - (\frac{x+x'}{2})]\cdot e = [\frac{x-x'}{2} + \alpha e]\cdot e = \frac{1}{2}\overbrace{(x-x')\cdot e}^{\leq 0} + \alpha\overbrace{e\cdot e}^{=0} \leq 0$$
$$[(x+\alpha e) - (\frac{x+x'}{2})]\cdot e' = [\frac{x-x'}{2} + \alpha e]\cdot e' = \frac{1}{2}\overbrace{(x-x')\cdot e'}^{\geq 0} + \overbrace{\alpha e\cdot e'}^{\geq 0} \geq 0$$

□

Corollary 2. *Let $S_{x,e}$ and $S_{x',e'}$ be non-parallel, future-directed lightlike strings. If $S_{x,e}$ and $S_{x'',e'}$ are causality disjoint for all x'' in some open neighborhood on x', then there exists a pair of wegdes W_L, W_R such that $W_L' = \overline{W_R}$, $W_R' = \overline{W_L}$, $S_{x,e} \subseteq W_L$ and $S_{x',e'} \subseteq W_R$.*

Proof. By the corollary 1, we have that

$$(x-x')^2 < 0 \ \wedge \ (x'-x)\cdot e > 0 \ \wedge \ (x-x').e' > 0 \tag{2.4}$$

Using theses inequalities one can show that the same wegdes of proposition 3 perform the separation $S_{x,e} \subseteq W_L$, $S_{x',e'} \subseteq W_R$. □

These results on wedge separation of strings will be applied in the proof of locality of the strings fields.

2.1.2 Useful lemmas

Lemma 1. *Let $e \in \mathbb{D}(\mathbb{M})$ be lightlike and future-directed and $\xi \in \mathbb{D}(\mathbb{M})$ arbitrary.*

(i) ξ *causal* $\wedge\ \xi \cdot e < 0 \Rightarrow \xi$ *past-directed*

(ii) ξ *causal* $\wedge\ \xi \cdot e > 0 \Rightarrow \xi$ *future-directed*

(iii) $\xi \cdot e = 0 \ \wedge \ \{e,\xi\} L.I. \Rightarrow \xi\cdot\xi < 0$

Lemma 2. *Let $v, w \in \mathbb{D}(\mathbb{M})$. Then,*

(i) $v^2, w^2 \geq 0$ *and* v, w *future-directed* $\Rightarrow v \cdot w \geq 0$

(ii) $v^2 = 0$ *and* $v \cdot w = 0 \Rightarrow w^2 \leq 0$

(iii) $v^2 = 0$, $w^2 \geq 0$ *and* $v \cdot w = 0 \Rightarrow w^2 = 0$ *e* $v \parallel w$,
where $v \parallel w :\Leftrightarrow \exists \kappa \in \mathbb{R}$ *s.t.* $v = \kappa w$

The proofs of the lemmas are found in the Appendix.

2.1.3 Time ordering of strings

Proposition 4. *Let $y \in \mathbb{M}$ and $S_{x,e}$ be a future-directed lightlike string. Then,*

$$y \succeq S_{x,e} \Leftrightarrow (y-x)\cdot e \geq 0 \ \wedge \ y \notin R_{x,e}$$

where $R_{x,e} := x + \mathbb{R}e$.

Proof. Let's prove both implications:

($\Rightarrow$) Contrapositive: $(y-x)\cdot e < 0 \vee y \in R_{x,e} \Rightarrow \neg(y \succeq S_{x,e})$

One has three cases:

(I) $(y-x)\cdot e < 0 \quad \wedge \quad (y-x)^2 \geq 0$

Applying item (i) of Lemma 1 for $\xi = y - x$, one gets

$$y - x \text{ past-directed } \Rightarrow y \in \overline{V_-(x)} \Rightarrow \neg(y \succeq x) \Rightarrow \neg(y \succeq S_{x,e})$$

(II) $(y-x)\cdot e < 0 \quad \wedge \quad (y-x)^2 < 0$

For $\alpha \geq \frac{(y-x)^2}{2(y-x)\cdot e} > 0$, one has

- $[y-(x+\alpha e)]^2 = (y-x)^2 - 2\alpha(y-x)\cdot e + \alpha^2 \cancelto{0}{e\cdot e} \geq (y-x)^2 - 2\left[\frac{(y-x)^2}{2(y-x)\cdot e}\right](y-x)\cdot e = 0$
- $[y-(x+\alpha e)]\cdot e = \underbrace{(y-x)\cdot e}_{<0} - \alpha \cancelto{0}{e\cdot e} < 0$

Thus, by item (i) of Lemma 1 for $\xi = y - (x+\alpha e)$, one arrives at

$$y - (x+\alpha e) \text{ past-directed } \Rightarrow y \in \overline{V_-(x+\alpha e)} \Rightarrow \neg(y \succeq x + \alpha e) \Rightarrow \neg(y \succeq S_{x,e})$$

(III) $y \in R_{x,e} = S_{x,e} \cup (R_{x,e} \setminus S_{x,e})$

- $y \in S_{x,e} \Rightarrow \neg(y \succeq S_{x,e})$
- $y \in R_{x,e} \setminus S_{x,e} \Rightarrow y - x \text{ past-directed} \quad \wedge \quad (y-x)^2 = 0 \Rightarrow y \in \overline{V_-(x)} \Rightarrow \neg(y \succeq S_{x,e})$

($\Leftarrow$) $(y-x)\cdot e \geq 0 \ \wedge \ y \notin R_{x,e} \Rightarrow y \succeq S_{x,e}$

Dividing in two cases:

(I) $(y-x)\cdot e = 0 \ \wedge \ y \notin R_{x,e}$

Applying item (iii) from Lemma 1 for $\xi = y - x$, one has that $(y-x)^2 < 0$. Now take $\alpha \in \mathbb{R}_0^+$.

$$[y-(x+\alpha e)]^2 = (y-x)^2 - 2\alpha \cancelto{0}{(y-x)\cdot e} + \alpha^2 \cancelto{0}{e\cdot e} < 0 \Rightarrow$$

$$\Rightarrow \forall \alpha \in \mathbb{R}_0^+ : y \in \mathbb{M} \setminus \overline{V_-(x+\alpha e)} \Rightarrow \forall \alpha \in \mathbb{R}_0^+ : y \succeq x + \alpha e \Rightarrow y \succeq S_{x,e}$$

(II) $(y-x)\cdot e > 0 \ \wedge \ y \notin R_{x,e}$

- (II.1) $(y-x)\cdot e > 0 \ \wedge \ (y-x)^2 < 0$

$$\forall \alpha \in \mathbb{R}_0^+ : [y-(x+\alpha e)]^2 = \underbrace{(y-x)^2}_{<0} \underbrace{-2\alpha(y-x)\cdot e}_{\leq 0} + \alpha^2 \cancelto{0}{e\cdot e} < 0 \Rightarrow$$

$$\Rightarrow \forall \alpha \in \mathbb{R}_0^+ : y \in \mathbb{M} \setminus \overline{V_-(x+\alpha e)} \Rightarrow y \succeq S_{x,e}$$

- (II.2) $(y-x)\cdot e > 0 \ \wedge \ (y-x)^2 \geq 0$

$$\forall \alpha \in \mathbb{R}_0^+ \text{ s.t. } 0 \leq \alpha \leq \frac{(y-x)^2}{2(y-x)\cdot e} : [y-(x+\alpha e)]^2 \geq 0 \ \wedge \ [y-(x+\alpha e)]\cdot e > 0$$

Using item (ii) from Lemma 1, one gets

$$y-(x+\alpha e) \text{ future-directed } \Rightarrow y \in \overline{V_+(x+\alpha e)} \subseteq \mathbb{M} \setminus \overline{V_-(x+\alpha e)}$$

$$\forall \alpha \in \mathbb{R}_0^+ \text{ t.q. } \alpha > \frac{(y-x)^2}{2(y-x)\cdot e} : [y-(x+\alpha e)]^2 < 0 \Rightarrow y \in \mathbb{M} \setminus \overline{V_-(x+\alpha e)}$$

Thus,

$$\forall \alpha \in \mathbb{R}_0^+ : y \in \mathbb{M} \setminus \overline{V_-(x+\alpha e)} \Rightarrow y \succeq S_{x,e}$$

This completes the proof. □

Proposition 5. *Let $S_{x,e}$ and $S_{x',e'}$ be future-directed lightlike strings. Then,*

$$S_{x',e'} \succeq S_{x,e} \Leftrightarrow (x'-x)\cdot e \geq 0 \ \wedge \ S_{x',e'} \cap R_{x,e} = \emptyset$$

Proof.

$$S_{x',e'} \succeq S_{x,e} \Leftrightarrow \forall y \in S_{x',e'} : y \succeq S_{x,e}$$

Using proposition 4, this is equivalent to

$$\forall y \in S_{x',e'} : (y-x)\cdot e \geq 0 \wedge y \notin R_{x,e} \Leftrightarrow \left(\forall \alpha \in \mathbb{R}_0^+ : \underbrace{[(x'+\alpha e')-x]\cdot e \geq 0}_{(i)} \right) \wedge S_{x',e'} \cap R_{x,e} = \emptyset$$

(i) $\forall \alpha \in \mathbb{R}_0^+ : [(x'+\alpha e')-x]\cdot e \geq 0 \Leftrightarrow$

$$\Leftrightarrow \forall \alpha \in \mathbb{R}_0^+ : (x'-x)\cdot e + \overbrace{\alpha e \cdot e'}^{\geq 0} \geq 0 \Leftrightarrow (x'-x)\cdot e \geq 0$$

Thus,

$$\forall\alpha \in \mathbb{R}_0^+ : [(x' + \alpha e') - x] \cdot e \geq 0 \ \wedge \ S_{x',e'} \cap R_{x,e} = \emptyset \Leftrightarrow$$
$$\Leftrightarrow (x' - x) \cdot e \geq 0 \ \wedge \ S_{x',e'} \cap R_{x,e} = \emptyset$$

□

This last proposition can be employed in the future to the construction of the time-ordered products of fields necessary for the study of interacting theories.

2.1.4 Secondary results: Chronological future and past, and causal completeness

Proposition 6. *Let $S_{x,e}$ be a future-directed lightlike string. Then,*

(i) $V_+(S_{x,e}) = V_+(x)$

(ii) $V_-(S_{x,e}) = \{y \in \mathbb{M} | (y - x).e < 0\}$

Proof. (i) Let's prove both inclusions:

($\supseteq$)

$$V_+(S_{x,e}) := \bigcup_{y \in S_{x,e}} V_+(y) = \bigcup_{\alpha \in \mathbb{R}_0^+} V_+(x + \alpha e) \supseteq V_+(x)$$

($\subseteq$)

$$y \in V_+(S_{x,e}) \Rightarrow \exists\alpha \in \mathbb{R}_0^+ : y \in V_+(x + \alpha e)$$

One has to look at two cases:

- $\alpha = 0 \Rightarrow y \in V_+(x)$
- $\alpha \neq 0 \Rightarrow y \in V_+(x + \alpha e) \Rightarrow [y - (x + \alpha e)]^2 > 0 \ \wedge \ [y - (x + \alpha e)] \cdot e > 0$
  That gives,

  $$0 < [y - (x + \alpha e)] \cdot e = (y - x) \cdot e$$
  $$0 < [y - (x + \alpha e)]^2 = (y - x)^2 - 2\alpha(y - x) \cdot e \Rightarrow (y - x)^2 > 2\alpha(y - x) \cdot e > 0$$

  Applying item (ii) from Lemma 1 for $\xi = y - x$, one arrives at $y \in V_+(x)$.

(ii) Let's prove both inclusions:

($\supseteq$)

Let $y \in \mathbb{M}$ such that $(y - x).e < 0$. One has two cases:

- $(y-x) \cdot e < 0 \ \wedge \ (y-x)^2 \geq 0$

  Using item (i) from Lemma 1 for $\xi = y - x$, one arrives at

$$y \in \overline{V_-(x)} \Rightarrow \forall \alpha \in \mathbb{R}_0^+ : V_-(x + \alpha e) \Rightarrow y \in V_-(S_{x,e})$$

- $(y-x) \cdot e < 0 \ \wedge \ (y-x)^2 < 0$
  Taking $\alpha > \frac{(y-x)^2}{2(y-x)\cdot e} > 0$, one has

$$[y - (x + \alpha e)]^2 = \underbrace{(y-x)^2 - 2\alpha(y-x)\cdot e}_{>0} + \alpha^2 \underbrace{e \cdot e}_{=0} > 0$$

$$[y - (x + \alpha e)] \cdot e = \underbrace{(y-x)\cdot e}_{<0} - \alpha \underbrace{e \cdot e}_{=0} < 0$$

  Using item (i) from Lemma 1 fro $\xi = y - (x + \alpha e)$, one gets

$$y \in V_-(x + \alpha e) \Rightarrow y \in V_-(S_{x,e})$$

$(\subseteq)$

$$y \in V_-(S_{x,e}) \Rightarrow \exists \alpha \in \mathbb{R}_0^+ : y \in V_-(x + \alpha e) \Rightarrow 0 > [y - (x + \alpha e)] \cdot e = (y - x) \cdot e$$

□

Proposition 7. *Let $S_{x,e}$ be a future-directed lightlike string. Then, $S_{x,e}$ is causally complete, that is, $S_{x,e} = (S_{x,e})''$*

Proof. One has that $S_{x,e} = (S_{x,e})'' \Leftrightarrow S_{x,e} \subseteq (S_{x,e})'' \ \wedge \ (S_{x,e})'' \subseteq S_{x,e}$.

The logical statement $S_{x,e} \subseteq (S_{x,e})''$ is a tautology that follows from the definition of causal complement [7]. Thus, it only remains to prove: $(S_{x,e})'' \subseteq S_{x,e}$.

$$(S_{x,e})'' \subseteq S_{x,e} :\Leftrightarrow \forall y \in \mathbb{M} : (y \in (S_{x,e})'' \Rightarrow y \in S_{x,e}) \underset{contrapositive}{\Leftrightarrow}$$
$$\Leftrightarrow \forall y \in \mathbb{M} : (y \notin S_{x,e} \Rightarrow y \notin (S_{x,e})'')$$

The below assertion also follows from the definition of causal complement [7].

$$\forall R \subseteq \mathbb{M} : R' = \mathbb{M} \setminus \left(\overline{V_+(R)} \cup \overline{V_-(R)}\right)$$

Setting $R = (S_{x,e})'$, one gets

$$\forall y \in \mathbb{M} : \left(y \notin S_{x,e} \Rightarrow y \in \left(\overline{V_+(S'_{x,e})} \cup \overline{V_-(S'_{x,e})}\right)\right)$$

Let $y \in \mathbb{M} \setminus S_{x,e}$ and consider the following cases:

(I) $(y-x)\cdot e<0$

Take $z\in\mathbb{M}\setminus R_{x,e}$ such that $(z-x)\cdot e=0$ and notice that, $z\in(S_{x,e})'$ and

$$(y-z)\cdot e=[y+(x-x)-z]\cdot e=\underbrace{(y-x)\cdot e}_{<0}+\underbrace{(x-z)\cdot e}_{=0}<0$$

- (I.1) $(y-z)^2\geq 0$

  Using item (i) from Lemma 1 for $\xi=y-z$, one has

  $(y-z)^2\geq 0\wedge(y-z)\cdot e<0\Rightarrow(y-z)$ past-directed $\Rightarrow y\in\overline{V_-(z)}\subseteq\overline{V_-(S'_{x,e})}\Rightarrow y\notin(S_{x,e})''$

- (I.2) $(y-z)^2<0$

  Taking $\alpha\geq\frac{(y-z)^2}{2(y-z)\cdot e}>0$, one gets

  $$[y-(z+\alpha e)]^2=(y-z)^2-2\alpha(y-z)\cdot e\geq(y-z)^2-2\left(\frac{(y-z)^2}{2(y-z)\cdot e}\right)=0$$
  $$[y-(z+\alpha e)]\cdot e=\underbrace{(y-z)\cdot e}_{<0}-\alpha\underbrace{e\cdot e}_{=0}<0$$

  and using, once more, item (i) from Lemma 1 from $\xi=y-(z+\alpha e)$, one arrives at

  $$y-(z+\alpha e)\text{ past-directed }\Rightarrow y\in\overline{V_-(z+\alpha e)}\subseteq\overline{V_-(S'_{x,e})}\Rightarrow y\notin(S_{x,e})''$$

(II) $(y-x)\cdot e>0$

Take $z\in\mathbb{M}\setminus R_{x,e}$ such that $(z-x)\cdot e=0$ and notice that, $z\in(S_{x,e})'$ and

$$(y-z)\cdot e=[y+(x-x)-z]\cdot e=\underbrace{(y-x)\cdot e}_{>0}+\underbrace{(x-z)\cdot e}_{=0}>0$$

- (II.1) $(y-z)^2\geq 0$

  Using item (ii) do Lemma 1 for $\xi=y-z$, one has

  $(y-z)^2\geq 0\ \wedge\ (y-z)\cdot e>0\Rightarrow(y-z)$ future-directed $\Rightarrow y\in\overline{V_+(z)}\subseteq\overline{V_+(S'_{x,e})}\Rightarrow$
  $\Rightarrow y\notin(S_{x,e})''$

- (II.2) $(y-z)^2<0$

Taking $\alpha = \frac{(y-z)^2}{2(y-z)\cdot e} < 0$, one gets

$$[y-(z+\alpha e)]^2 = (y-z)^2 - 2\alpha(y-z)\cdot e = (y-z)^2 - 2\left(\frac{(y-z)^2}{2(y-z)\cdot e}\right) = 0$$

$$[y-(z+\alpha e)]\cdot e = \underbrace{(y-z)\cdot e}_{>0} - \alpha \underbrace{e\cdot e}_{=0} > 0$$

and using again the item (ii) from Lemma 1 for $\xi = y-(z+\alpha e)$, one arrives at

$$y-(z+\alpha e) \text{ future-directed } \Rightarrow y \in \overline{V_+(z+\alpha e)} \subseteq \overline{V_+(S'_{x,e})} \Rightarrow y \notin (S_{x,e})''$$

(III) $y \notin S_{x,e} \ \wedge \ (y-x)\cdot e = 0$

- (III.1) $y \notin R_{x,e}$

  Straightfoward, since $\{y \in \mathbb{M} | y \notin R_{x,e} \ \wedge \ (y-x)\cdot e = 0\} \subseteq (S_{x,e})'$

- (III.2) $y \in S_{x,-e} \setminus \{x\}$

  Take $z \in (S_{x,e})'$ such $(z-x)\cdot e > 0 \ \wedge \ (z-y)^2 \geq 0$ and notice that

$$(y-z)\cdot e = \underbrace{(y-x)\cdot e}_{=0} + \underbrace{(x-z)\cdot e}_{<0} < 0$$

  Using item (i) from Lemma 1 for $\xi = y-z$, one arrives at

$$(y-z)^2 \geq 0 \wedge (y-z)\cdot e < 0 \Rightarrow (y-z) \text{ past-directed } \Rightarrow y \in \overline{V_-(z)} \subseteq \overline{V_-(S'_{x,e})} \Rightarrow y \notin (S_{x,e})''$$

□

2.2 Lightlike string-local fields

2.2.1 Considerations on wedges

First we see some properties related to Wedges which will be use in the proof of locality of the string-local fields.

Consider the wedge W_1,

$$W_1 \doteq \left\{x \in \mathbb{R}^4 |\ x^1 > \left|x^0\right|\right\}, \tag{2.5}$$

together with the one-parameter group $\Lambda_1(\cdot)$ of Lorentz boosts which leave W_1 invariant, and the reflection j_1 across the edge of the wedge. More specifically, $\Lambda_1(t)$ acts as

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

and j_1 acts as the reflection on the coordinates x^0 and x^1, leaving the other coordinates unchanged. As in equation (2.2), general wedge regions are the Poincaré transforms of W_1. For the wedge

$$W = \Lambda W_1 + a \text{ , with } (a, \Lambda) \in \mathcal{P}_+^\uparrow \tag{2.6}$$

one defines the corresponding boosts $\Lambda_W(\cdot)$ and reflection j_W by

$$\Lambda_W(t) := (a, \Lambda)\, \Lambda_1(t)\, (a, \Lambda)^{-1}, \quad j_W := (a, \Lambda)\, j_1\, (a, \Lambda)^{-1}. \tag{2.7}$$

(See [14] for well-definedness.) The reflection j_W results from analytic extension of the (entire analytic) $L_+(\mathbb{C})$-valued function $\Lambda_W(z)$ at $z = i\pi$, and j_1 also can be represented (in 4 dimensions) as the composition of the PT transformation with a π-rotation about the 1-axis, $j_1 = -R_1(\pi)$.

2.2.2 Remarks on the general construction of lightlike string-local free fields

As pointed out in section 1.2.3, we want to contruct free quantum fields for higher spin particles without giving up on fundamental properties, such as the Hilbert space positivity, Poincaré covariance and locality, and still get a renormalizable interacting theory. The alternative proposed was to introduce the so-called string-localized fields and in this dissertation we explore some aspects of lightlike string-localized fields, equations (1.73) and (1.72).

Let $\mathcal{H}$ be the Bosonic Fock space over the irreducible space $\mathcal{H}^{(m,s)}$ for single particles of mass m and spin s, U be the second quantization of the single particle representation $U^{(m,s)}$ and Ω the invariant Fock space vaccum. We want to construct here not only vector or tensor fields, but an N-component field $\varphi_r(x, e)$, $r = 1, \dots, N$, which transforms in a covariant way under some matrix representation D of the Lorentz group, i.e. Eq. (1.73) is replaced by

$$U(a, \Lambda)\, \varphi_r(x, e)\, U(a, \Lambda)^{-1} = \sum_{r'=1}^{N} \varphi_{r'}(a + \Lambda x, \Lambda e)\ D(\Lambda)_{r'r}. \tag{2.8}$$

and are covariant under the parity transformation $P : (x^0, x) \mapsto (x^0, -x)$, i.e., the representations U and D extend to the orthochronous Lorentz group and Eq. (2.8) also holds for $\Lambda = P$. We give a a brief sketch [1] on the contruction of lightlike string-local quantum fields from the point of view of intertwiners, along the lines of [4] (for fields localized on spacelike strings).

The idea is to construct objects that link (or intertwine) the transformation behaviour of the field $\varphi_r(x, e)$ (under the N-dimensional representation D) to the transformation behaviour of one-particle states (under the $2s + 1$-dimensional representation

[1] A complete description of the contruction is under current investigation [16]

$D^{(s)}$), which the field creates from the vaccum, i.e. for $p \in H_m^+$

$$\Big(\varphi_r(x,e)\Omega\Big)(p) := (2\pi)^{-\frac{3}{2}}\, e^{ip\cdot x}\, v_r(p,e) \quad \in \ \mathbb{C}^{2s+1}. \tag{2.9}$$

calling the family of $\mathbb{C}^{2s+1}$-valued distributions $v_r(p,e)$, the intertwiner of the field φ_r. Considering the case when the particle and anti-particle coincide,there is a $\mathbb{C}^{2s+1}$-valued distribution $v_r^c(p,e)$, the so-called conjugate intertwiner, such that for $p \in H_m^+$

$$\Big(\varphi_r(x,e)^*\Omega\Big)(p) := (2\pi)^{-\frac{3}{2}}\, e^{ip\cdot x}\, v_r^c(p,e) \quad \in \ \mathbb{C}^{2s+1}.. \tag{2.10}$$

We interpret the family $v_r(p,e)$ as a linear map from some N-dimensional vector space $\mathfrak{h}$ with basis $\{e_{(1)},\ldots,e_{(N)}\}$, the "target space" of the field, to $\mathbb{C}^{2s+1}$ via

$$v(p,e)\, e_{(r)} := v_r(p,e) \qquad \in \mathbb{C}^{2s+1}, \tag{2.11}$$

and the matrix $D(\Lambda)_{r'r}$, in (2.8), as a linear endomorphism of $\mathfrak{h}$. Now we give a precise definition of these intertwiners, making explicit the intertwining property and demanding a further property of analyticity.

Definition 5 (Intertwiners)**.** *A family of distributions v_r, $r = 1,\ldots,N$, is called a Wigner intertwiner from D to $D^{(s)}$ if*
i) it satisfies the intertwiner relation

$$D^{(s)}\Big(R(\Lambda,p)\Big) \circ v(\Lambda^{-1}p, \Lambda^{-1}e) = v(p,e) \circ D(\Lambda), \quad \Lambda \in \mathcal{L}^{\uparrow}. \tag{2.12}$$

ii) For almost all p and all e, the function $\mathbb{R}^+ \ni t \mapsto v_r(p,te) \in \mathbb{C}^{2s+1}$ is the boundary value of an analytic function $z \mapsto \tilde{v}_r(p,ze)$ on the upper complex half plane $\mathbb{R}+i\mathbb{R}^+$ and satisfies the following bound: There is a positive function M on H_m^+ which is locally L^2 w.r.t. $d\mu$ and polynomially bounded, and for every pair of compact subsets K of the upper complex half plane and $\Theta \subset H_0^+$ there is a constant $c = c_{K,\Theta}$ such that for all $(z,e) \in K\times\Theta$ holds

$$\|\tilde{v}_r(p,ze)\| \leq c\, M(p). \tag{2.13}$$

(The norm on the l.h.s. refers to the little Hilbert space $\mathbb{C}^{2s+1}$..)
(iii) Given a Wigner intertwiner v, its conjugate v^c is defined by

$$D^{(s)}(-\mathbb{1}\,) \circ v^c(p,e) \circ C = v(p,-e) \circ D(-\mathbb{1}\,). \tag{2.14}$$

Here, $v(p,-e)$ arises from $v(p,ze)$ via analytic continuation from $z=1$ to $z=-1$ through the upper half plane, and $D(-\mathbb{1}\,)$ arises from the unit component via analytic continuation through the complex proper Lorentz group $L_+(\mathbb{C})$.[2] Further, C is the anti-linear involution defined by

$$C\sum_{r=1}^{N} z_r\, e_{(r)} \doteq \sum_{r=1}^{N} \overline{z_r}\, e_{(r)}.$$

[2] Note that all finite dimensional representations D extend analytically to the complex Lorentz group $L_+(\mathbb{C})$, and that the latter is path connected. A path from $\mathbb{1}$ to $-\mathbb{1}$ is for example $\Lambda_1(is)$, $s \in [0,\pi]$, composed with $R_1(\alpha)$, $\alpha \in [0,\pi]$.

Note that $D(-\mathbb{1})$ is linear, while $D^{(s)}(-\mathbb{1})$ is anti-linear. This conjugate intertwiner v^c satisfies the same relation (2.12), but with $D(\Lambda)$ replaced by the componentwise complex conjugate $\overline{D(\Lambda)}$. v is called self-conjugate if $v = v^c$.

The condition (ii) involving analyticity properties of the intertwiners is motivated by the assumption that lightlike string fields should satisfy the so-called Bisognano-Wichmann property [17], which holds for spacelike string fields [18]. This condition will be used to prove string-localization.

One then constructs a free field via the Wigner intertwiners, as follows: Let $a^*(\psi)$ and $a(\psi)$, $\psi \in \mathcal{H}^{(m,s)}$, denote the creation and annihilation operators. Given $f \in \mathcal{S}(\mathbb{R}^4)$, define the following single particle vectors [3]

$$\psi_r(p) = (2\pi)^{-\frac{3}{2}} \hat{f}(p) v_r(p,e) \qquad \psi_r^c(p) = (2\pi)^{-\frac{3}{2}} \hat{\bar{f}}(p) v_r^c(p,e) \tag{2.16}$$

The free field for the Wigner intertwiner v is given by[4]

$$\varphi_r(f,e) := a^*(\psi_r) + a(\psi_r^c), \tag{2.17}$$

Let us re-write it in the usual informal notation

$$a^*(\psi) =: \int\limits_{H_m^+} d\mu_m(p) \sum_{k=1}^{2s+1} \psi^k(p) a^*(p,k) \qquad a(\psi) =: \int\limits_{H_m^+} d\mu_m(p) \sum_{k=1}^{2s+1} \overline{\psi^k(p)} a(p,k) \tag{2.18}$$

where the superscript k denotes the components with respect to a basis $\{e_{(k)}\}$ in $\mathbb{C}^{2s+1}$, ie, $\psi(p) = \sum_{k=1}^{2s+1} \psi^k(p) e_{(k)}$. Then [5]

$$\varphi_r(x,e) = (2\pi)^{-\frac{3}{2}} \int_{H_m^+} d\mu_m(p) \sum_{k=1}^{2s+1} \left\{ e^{ip\cdot x} v_r^k(p,e) a^*(p,k) + e^{-ip\cdot x} \overline{v_r{}^{c,k}(p,e)} a(p,k) \right\} \tag{2.20}$$

It is hermitean if and only if the intertwiner is self-conjugate, $v^c = v$.

The two-point function of two such fields $\varphi_{1,r}$ and $\varphi_{2,r}$ with respective Wigner intertwiners v_1 and v_2 comes out as

$$(\,\Omega, \varphi_{1,r}(x,e)\, \varphi_{2,r'}(x',e')\Omega\,) = (2\pi)^{-3} \int_{H_m^+} d\mu_m(p)\, e^{-ip\cdot(x-x')}\, M_{r,r'}^{\varphi_1\varphi_2}(p,e,e') \tag{2.21}$$

$$M_{r,r'}^{\varphi_1\varphi_2}(p,e,e') = (\, v_{1r}^c(p,e), v_{2r'}(p,e')\,)_{\mathbb{C}^{2s+1}}\,, \tag{2.22}$$

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[3] We adopt the following convention for the Fourier transform $\hat{f}$, whose inverse we denote by $\check{f}$:

$$\hat{f}(p) \doteq \int d^4x\, e^{ip\cdot x} f(x), \quad \check{f}(x) = (2\pi)^{-4} \int d^4p\, e^{-ip\cdot x} f(p). \tag{2.15}$$

[4] Considering the case when the anti-particle coincides with the particle

[5] Of course, rigorously speaking, $\varphi(x,e)$ is informally defined by

$$\varphi_r(f,e) = \int d^4x\, f(x)\, \varphi_r(x,e). \tag{2.19}$$

where $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\mathbb{C}^{2s+1}}$ denote the scalar products in Fock space $\mathcal{H}$ and in the little Hilbert space $\mathbb{C}^{2s+1}$, respectively. $M^{\varphi_1\varphi_2}_{r,r'}(p,e,e')$ is called the on-shell part of the two-point function. Note that positivity of the two-point function is satisfied by construction[11]. Another important remark is that this construction holds for point-local fields as well by symply neglecting the e-variable.

Proposition 8. *Let $v(p,e)$ be a Wigner intertwiner from $D^{(s)}$ to D in the sense of Definition 5, and let $v^c(p,e)$ be defined by Eq. (2.14). Then the field defined in Eq. (2.20) is a distribution in the weak sense.*[6] *It is string-localized and covariant in the sense of Eqs. (1.72) and (2.8), furthermore, the CPT symmetry holds:*

$$U(-\mathbb{1})\,\varphi_r(x,e)\,U(-\mathbb{1})^{-1} = \sum_{r'=1}^{N} \varphi_{r'}(-x,-e)^* D(-1)_{r'r}. \tag{2.23}$$

Proof. The (weak) distribution property, covariance (2.8) and CPT symmetry (2.23) are shown as in [4]. The proof of string-locality (1.72) is analogous to the proof for spacelike strings in [4], using the appropriated results for wegde separation and analyticity of the lightlike string fields. Therefore, we quote the proof with the modifications related to the lightlike causal character.

Suppose, $S_{x,e}$ and $S_{x',e'}$ satisfy the condition for Eq. (1.72), ie $S_{x,e}$ and $S_{x'',e'}$ are causally disjoint for all x'' in an open neighborhood of x'. This implies, by corollary 2, that there is a wedge region W such that $S_{x,e} \subset W$ and $S_{x',e'} \subset \mathring{W}'$, where $\mathring{W}'$ denotes the interior of the causal complement of W. Let j_W and $\Lambda_W(t)$ be the reflection and the boosts, respectively, corresponding to W, i.e., $j_W \doteq (a,\Lambda)\, j_1\,(a,\Lambda)^{-1}$ and $\Lambda_W(t) \doteq (a,\Lambda)\Lambda_1(t)(a,\Lambda)^{-1}$ if $W = (a,\Lambda)W_1 \equiv a+\Lambda W_1$. Denote by g_t the proper non-orthochronous Poincaré transformation $\Lambda_W(-t)j_W$. Then one verifies the relation

$$\begin{aligned}\Big(\Omega, \varphi_r(x,e)^*\varphi_{r'}(g_t^{-1}x', g_t^{-1}e')\Omega\Big) \\ = \sum_{s,s'} (\,\Omega, \varphi_{s'}(x',e')\varphi_s(g_t x, g_t e)^*\Omega\,)\ \overline{D(g_t)_{sr}}\, D(g_t)_{s'r'}.\end{aligned} \tag{2.24}$$

(We have successively used invariance of Ω under $U \equiv U(g_t)$, anti-unitarity of U, namely $(\,\Omega,\psi\,) = (\,U^{-1}\Omega,\psi\,) = (\,U\psi,\Omega\,)$, and then covariance (2.8) and the CPT symmetry (2.23). Finally we have adjoined the field operators to the right hand side of the scalar product.) The matrix-valued function $D(g_t)$ (and hence $\overline{D(g_{\bar{t}})}$) is entire analytic in the boost parameter t. Note that j_W and $\Lambda_W(t)$ commute, hence $g_t^{-1} = g_{-t}$, and that for t in the strip $\mathbb{R} + i(0,\pi)$ the imaginary parts of $g_t x$ and $g_{-t}x'$ lie in the forward light cone V_+ and the imaginary parts of $g_t e$, and $g_{-t}e'$ lie in the closure of the forward light cone, $\overline{V_+}$ (see for example Eq. (A.7) in [3]). Now the two-point function is an analytic function of the second x-variable in the tube $\mathbb{R}^4 + iV_+$ due to the support of its Fourier transform, and

[6] More precisely, its matrix elements between state vectors with finite particle numbers are distributions.

also of the second e-variable due to the analyticity of the intertwiner function. Therefore Eq. (2.24) extends, by the Schwarz reflection principle, from the boundary at $\operatorname{Im} t = 0$ over the entire strip to the upper boundary, and the boundary values at $t = i\pi$ coincide for both sides. But $\Lambda_W(\pm i\pi) = j_W$, i.e., $g_{\pm i\pi} = 1$, and thus Eq. (2.24) at $t = i\pi$ is just the locality of the two-point functions. This implies locality of the fields by the usual Jost-Schroer arguments.

□

2.2.3 String-local fields as line integrals over point-local fields

Following [4], all string-localized fields we are treating in this dissertation can be viewed as n-fold line integrals over point fields. These integrals are to be understood in the following sense. Let $n \in \mathbb{N}$ and let $\varphi^p(x)$ be some free point-local field. (It may be a tensor field, but we omit the tensor indices). Then for fixed $e \in H_0^+$ and $\nu \in \mathbb{N}$ define

$$\varphi_{(\nu)}(x,e) := \int_0^\nu ds_1 \cdots \int_0^\nu ds_n\, \varphi^p\Big(x + (s_1 + \cdots + s_n)e\Big) \tag{2.25}$$

It exists as a distribution in x and a function in e. The point-local field $\varphi^p(x)$ creates bosons of spin s and mass m, furthermore it has intertwiner function v^p, which satisfies a similar relation to (2.12) with the e-variable neglected, ie

$$D^{(s)}\Big(R(\Lambda,p)\Big) \circ v^p(\Lambda^{-1}p) = v^p(p) \circ D(\Lambda), \quad \Lambda \in \mathcal{L}^\uparrow. \tag{2.26}$$

Now we want to see if in the limit $\nu \to \infty$, $\varphi_{(\nu)}(x,e)$ gives a lightlike string-local field.

Lemma 3. *l Let $\varphi_{(\nu)}(x,e)$ be the field given by (2.25). Then*

(i) the intertwiner of $\varphi_{(\nu)}(x,e)$ is

$$v_\nu(p,e) = \frac{i^n}{(p\cdot e)^n}(1 - e^{i\nu p\cdot e})^n\, v^p(p) \tag{2.27}$$

(ii) the field $\varphi_{(\nu)}(x,e)$, as $\nu \to \infty$, converges to the covariant lightlike string-local field $\varphi(x,e)$, defined by its intertwiner

$$v(p,e) = \frac{i^n}{(p\cdot e)^n}\, v^p(p), \tag{2.28}$$

in the sense that the two point-funtion

$$w_\nu(x - x', e, e') = (\Omega, \varphi(x,e)\varphi_{(\nu)}(x',e')\Omega) \tag{2.29}$$

converges in $\mathcal{D}'(\mathbb{R}^4)$ [7] to

$$w(x - x', e, e') = (\Omega, \varphi(x,e)\varphi(x',e')\Omega) \tag{2.30}$$

[7] $\mathcal{D}'(\mathbb{R}^4)$ is the space of distributions over $\mathcal{D}(\mathbb{R}^4)$

The proof is found in the appendix. With the result (ii) one can prove [11] that for all $n \in \mathbb{N}_0$,

$$(\Phi, \varphi_{(\nu)}(x,e)\Psi) \stackrel{\nu\to\infty}{\longrightarrow} (\Phi, \varphi(x,e)\Psi)$$

for all $\Phi, \Psi \in \mathcal{D}_n := span\{\varphi(x_1,e)\varphi(x_2,e)\cdots\varphi(x_n,e)\Omega\}$. Hence we will denote the field $\varphi(x,e)$ by

$$\varphi(x,e) \equiv \int_0^\infty ds_1 \cdots \int_0^\infty ds_n\, \varphi^p\Big(x + (s_1 + \cdots s_n)e\Big). \tag{2.31}$$

and its intertwiner by

$$v(p,e) = t^n(p,e)\; v^p(p). \tag{2.32}$$

where $t^n(p,e) := \frac{i^n}{(p\cdot e)^n}$.

For later reference, we exhibit the two-point function of these line integrals. Let $\varphi_1^p(x)$ and $\varphi_2^p(x)$ be free point-local fields for the same particle type, let $M^p(p)$ be the on-shell part of its two-point function, which is a polynomial [8]. Let, for $i = 1$ and 2, $\varphi_i(x,e)$ be the string-localized field constructed from $\varphi_1^p(x)$ by an n_i-fold line integral as in Eq. (2.31). Recalling Eq. (2.32), the on-shell part of the corresponding two-point function $w(x-x',e,e') \doteq (\,\Omega, \varphi_1(x,e)\varphi_2(x',e')\Omega\,)$ is then given by

$$M(p,e,e') = \frac{i^{n_2-n_1}\, M^p(p)}{(p\cdot e)^{n_1}(p\cdot e')^{n_2}}. \tag{2.33}$$

2.3 Scaling degree of the two-point functions

Consider the following lightlike string-local fields:

$$\varphi_i(x,e) = \int_0^\infty ds_1 \cdots \int_0^\infty ds_{n_i} \varphi^P(x + (s_1 + \cdots + s_{n_i})e), \text{ i=1,2}$$

where φ^P is a point-local field with the on shell part of its two-point function, M^P, being a polynomial of degree d.

The two-point function $w_m(x-x',e,e') = (\Omega, \varphi_1(x,e)\varphi_2(x',e')\Omega)$, labeled by the mass $m > 0$, is given by

$$w_m(x-x',e,e') = (2\pi)^{-3} \int_{H_m^+} d\mu_m(p) e^{-ip\cdot(x-x')}\overline{t^{n_1}(p,e)}t^{n_2}(p,e')M^P(p) \tag{2.34}$$

Although $w_m(x-x',e,e')$ is a function of the e-variables, for the calculation of the scaling degree, we want to consider the smearing [8] also in e and e', i.e. see $t^n(p,e)$ as a distribution

[8] ie the procedure of multiplying by a suitable function and integrating with respect that function variable

in $\mathcal{D}'(H_0^+)$ [9]. Hence, after the smearing with $f \in \mathcal{D}(\mathbb{R}^4)$ and $h, h' \in \mathcal{D}(H_0^+)$ we have

$$w_m(f,h,h') \equiv \int d^4\xi f(\xi) \int_{H_0^+} d\sigma(e)h(e) \int_{H_0^+} d\sigma(e')h'(e')w_m(\xi,e,e') = \tag{2.35}$$

$$= 2\pi \int_{H_m^+} d\mu_m(p)\check{f}(p)M^P(p)\overline{t^{n_1}(p,h)}t^{n_2}(p,h') \tag{2.36}$$

where for $p \in H_m^+$ e $h \in \mathcal{D}(H_0^+)$,

$$t^n(p,h) := \int_{H_0^+} d\sigma(e)h(e)\frac{i^n}{(p\cdot e)^n} \tag{2.37}$$

The following lemma is going to be used to calculate the scaling degree of w_m.

Lemma 4. *Given $p \in H_m^+$ and $h \in \mathcal{D}(H_0^+)$, consider $t^n(p,h)$ defined by (2.37). Then, one has that*

$$t^n(p,h) = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \tilde{h}\left((s_1 + \cdots s_n)p)\right) \tag{2.38}$$

where

$$\tilde{h}(p) := \int_{H_0^+} d\sigma(e)h(e)e^{i(p\cdot e)} \tag{2.39}$$

and furthermore, there exists a constant $K \in \mathbb{R}$ such that, for all $p \in H_m^+$, the following bound holds

$$|t^n(p,h)| \le \frac{K}{{p_0}^n} \tag{2.40}$$

The proof is found in the appendix.

Now we determine the UV behaviour of the two-point function w_m defined above by calculating its scaling degree.

Definition 6. *Let $u(\xi,e,e')$ be a distribution on $\mathbb{R}^4 \times H_0^+ \times H_0^+$. We define the scaling degree of u, denoted by $sd(u)$, with respect to ξ after smearing in e, e' as*

$$\underset{\omega\in\mathbb{R}}{inf}\left\{\lambda^\omega\langle u_\lambda, f\otimes h\otimes h'\rangle \xrightarrow{\lambda\to 0} 0\right\} \text{ for all } f \in D(\mathbb{R}^4), h, h' \in \mathcal{D}(H_0^+) \tag{2.41}$$

where the rescaled distribution u_λ is defined as

$$\langle u_\lambda, f\otimes h\otimes h'\rangle := \langle u, f^\lambda \otimes h \otimes h'\rangle \text{ with } f^\lambda(\xi) := \lambda^{-4}f(\lambda^{-1}\xi)$$

Proposition 9. *Let $w_m(\xi,e,e')$ be the two-point function given by (2.35), then the scaling degree of w_m after smearing in the e-variables is the maximum of 0 and $d + 2 - (n_1 + n_2)$*

[9] Note that $H_0^+ \equiv H_{m=0}^+ \overset{!}{=} \partial V_+ \setminus \{0\}$

Proof. First assume that M^P is homogeneous of degree d. Let $f \in \mathcal{D}(\mathbb{R}^4)$, $h, h' \in \mathcal{D}(H_0^+)$ and consider the rescaled two-point function

$$w_m^\lambda(f,h,h') = 2\pi \int\limits_{H_m^+} d\mu_m(p)\check{f}(\lambda p)M^P(p)\overline{t^{n_1}(p,h)}t^{n_2}(p,h') \tag{2.42}$$

where has been used that $\check{f}^\lambda(p) = \check{f}(\lambda p)$. Performing the change of variables $\lambda p \to p$ and using $d\mu_m(\lambda^{-1}p) = \lambda^{-2}d\mu_{\lambda m}(p)$, one arrives at

$$w_m^\lambda(f,h,h') = 2\pi\lambda^{-d-2+(n_1+n_2)} \int\limits_{H_{\lambda m}^+} d\mu_{\lambda m}(p)\check{f}(p)M^P(p)\overline{t^{n_1}(p,h)}t^{n_2}(p,h')$$

Using that $|M^P(p)| \leq C_0\left(\omega_{\lambda m}(\vec{p})\right)^d$, $|t^{n_i}(p,h)| \leq N\left(\omega_{\lambda m}(\vec{p})\right)^{-n_i}$ (lemma 4), $|\check{f}(p)| \leq C(1+|\vec{p}|)^{-r}$ and setting $n = n_1 + n_2$ one has

$$|w_m^\lambda(f,h,h')| \leq 2\pi\lambda^{-d-2+n} \int d^3\vec{p}\varphi(\vec{p})\left(\omega_{\lambda m}(\vec{p})\right)^{d-n-1}$$

where $\varphi(\vec{p}) := NC_0C(1+|\vec{p}|)^{-r}$ is a λm-independent $C^\infty(\mathbb{R}^3)$ function that decreases more rapidly than $|\vec{p}|^r$. Using that

$$\frac{1}{\left(\omega_{\lambda m}\right)^2} \leq \frac{1}{|\vec{p}|^2}$$

we can rewrite the above expression as

$$|w_m^\lambda(f,h,h')| \leq 2\pi\lambda^{-d-2+n} \int d^3\vec{p}\varphi(\vec{p})\left(\omega_{\lambda m}(\vec{p})\right)^{d-n+1}|\vec{p}|^{-2} \tag{2.43}$$

- Consider $d-n+1 \geq 0$.

  For $\lambda \in [0,1]$, it holds that $\omega_{\lambda m}(\vec{p}) \leq \omega_m(\vec{p})$ a, consequently, for $\varepsilon > 0$

  $$|\lambda^{(d+2-n)+\varepsilon}w_m^\lambda(f,h,h')| \leq 2\pi\lambda^\varepsilon \underbrace{\int d^3\vec{p}\varphi(\vec{p})\left(\omega_m(\vec{p})\right)^{d-n+1}|\vec{p}|^{-2}}_{<\infty} \xrightarrow{\lambda\to 0} 0^{10}$$

  Hence

  $$sd(w_m) = d+2-n$$

- Consider $d-n+1 < 0$.

  The fact that $\omega_m(\kappa) \geq m$ and $\omega_m(\kappa) \geq \frac{\kappa+m}{\sqrt{2}}$, supplies us with the following bound

  $$\left(\omega_{\lambda m}(\kappa)\right)^{d-n+1} \leq \sqrt{2}\frac{(\lambda m)^{d-n+2}}{\kappa+\lambda m}$$

[10] Taking r such that $r-(d-n+1) > 1$, which is always possible, makes the integral convergent

Considering $\lambda \in [0,1]$ and subistituing the above bound in w_m^λ one has that, for $\varepsilon > 0$:

$$|\lambda^{0+\varepsilon} w_m^\lambda(f,h,h')| \leq \lambda^\varepsilon 8\pi^2\sqrt{2}m^{d-n+2} \int_0^\infty d\kappa \; \frac{\varphi(\kappa)}{k+\lambda m} =$$

$$= \lambda^\varepsilon 8\pi^2\sqrt{2}m^{d-n+2} \left[\underbrace{\int_0^m d\kappa \; \frac{\varphi(\kappa)}{k+\lambda m}}_{(I)} + \underbrace{\int_m^\infty d\kappa \; \frac{\varphi(\kappa)}{k+\lambda m}}_{(II)} \right]$$

Noticing that

(I) $\int_0^m d\kappa \; \frac{\varphi(\kappa)}{k+\lambda m} \leq C_1 \int_0^m \frac{d\kappa}{k+\lambda m} = C_1 \; ln(\frac{1+\lambda}{\lambda}) \leq C_1 \; ln(\frac{2}{\lambda})$

(II) $\int_m^\infty d\kappa \; \frac{\varphi(\kappa)}{k+\lambda m} \leq \int_m^\infty d\kappa \; \frac{\varphi(\kappa)}{k} \leq C \int_m^\infty \frac{d\kappa}{\kappa^{r+1}} = C_2 < \infty$

One arrives at

$$|\lambda^\varepsilon w_m^\lambda(f,h,h')| \leq \lambda^\varepsilon 8\pi^2\sqrt{2}m^{d-n+2} \left[C_1\lambda^\varepsilon \; ln(\frac{2}{\lambda}) + C_2\lambda^\varepsilon \right] \xrightarrow{\lambda\to 0} 0$$

Hence,

$$sd(w_m) = 0$$

If the polinomial M^P is not homogeneous, the lower degree monomials will contribute with terms yielding lower scaling degrees by the same token. □

One sees that, by choosing the number of integrations n_i, the scaling dimension (half of the scaling degree) of the initial fields φ_i^P, namely $d+2$, can be lowered to zero. Thus one has a better perspective on building renormalizable interactions.

Conclusion

Our results on the properties of lightlike string-local quantum show that these are indeed a very interesting alternative relative to both the usual point-local fields and the spacelike string-local fields. With respect to point-local fields, the lightlike string fields studied have, just as spacelike ones, the desired advantage of the good UV behaviour without giving up the fundamental assumptions of positivity, covariance and locality. Furthermore, when comparing them to the spacelike ones, we note that they have simpler analytical expressions, they are functions with respect to the e-variable, and analyticity properties (definition 5, item (ii)). The future perpectives consist in give a complete characterization of these fields, in the point of view of Wigner intertwiners, and embark on the construction of interacting models [4, Outlook].

Appendix

APPENDIX A – Proof of the Lemmas

A.1 Proof of the Lemmas

Proof of Lemma 1. (i) Let e be lightlike and future-directed and ξ causal ($\xi \cdot \xi \geq 0$). Let's prove the contrapositive

$$(\xi \cdot e < 0 \Rightarrow \xi \text{ past-directed}) \underset{\text{contrapositive}}{\Leftrightarrow} (\xi^0 \geq 0 \Rightarrow \xi \cdot e \geq 0)$$

One has that

- $\xi^0 \geq 0 \ \wedge \ \xi \cdot \xi \geq 0 \Rightarrow (\xi^0)^2 \geq |\vec{\xi}|^2 \Rightarrow \xi^0 \geq |\vec{\xi}|$
- $e^0 > 0 \ \wedge \ e \cdot e = 0 \Rightarrow (e^0)^2 = |\vec{e}|^2 \Rightarrow e^0 = |\vec{e}|$

Therefore,

$$\xi^0 e^0 = \xi|\vec{e}| \geq |\vec{\xi}||\vec{e}| \overset{CS}{\geq} |\vec{\xi} \cdot \vec{e}| \geq \vec{\xi} \cdot \vec{e} \Rightarrow \xi \cdot e \geq 0^1$$

(ii) Let e be lightlike and future-directed and ξ causal ($\xi \cdot \xi \geq 0$). Let's prove the contrapositive

$$(\xi \cdot e > 0 \Rightarrow \xi \text{ futuro}) \underset{\text{contrapositive}}{\Leftrightarrow} (\xi^0 < 0 \Rightarrow \xi \cdot e \leq 0)$$

One has that

- $\xi^0 < 0 \ \wedge \ \xi \cdot \xi \geq 0 \Rightarrow (\xi^0)^2 \geq |\vec{\xi}|^2 \Rightarrow \xi^0 \leq -|\vec{\xi}|$
- $e^0 > 0 \ \wedge \ e \cdot e = 0 \Rightarrow (e^0)^2 = |\vec{e}|^2 \Rightarrow e^0 = |\vec{e}|$

Therefore,

$$\xi^0 e^0 = \xi|\vec{e}| \leq -|\vec{\xi}||\vec{e}| \overset{CS}{\leq} -|\vec{\xi} \cdot \vec{e}| \leq \vec{\xi} \cdot \vec{e} \Rightarrow \xi \cdot e \leq 0$$

(iii) Let e be lightlike and future-directed and $\{e, \xi\}$ L.I.

$$0 = \xi^0 e^0 - \vec{\xi} \cdot \vec{e} \underset{e^0>0}{\Rightarrow} \xi^0 = \frac{\vec{\xi} \cdot \vec{e}}{e^0}$$

$$\xi \cdot \xi = (\xi^0)^2 - |\vec{\xi}|^2 = \frac{|\vec{\xi} \cdot \vec{e}|^2}{(e^0)^2} - |\vec{\xi}|^2 \overset{CS}{<} \frac{|\vec{\xi}|^2|\vec{e}|^2}{(e^0)^2} - |\vec{\xi}|^2 = 0 \Rightarrow \xi \cdot \xi < 0$$

□

Proof of Lemma 2. (i) One has that

[1] CS means that we used the Cauchy-Schwarz inequality for the spatial part of the inner product,ie for $\vec{u}, \vec{v} \in \mathbb{R}^3$, $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$

- $v^2 \geq 0 \Rightarrow (v^0)^2 \geq |\vec{v}|^2 \Rightarrow |v^0| \geq |\vec{v}| \overset{v^0 \geq 0}{\Rightarrow} v^0 \geq |\vec{v}|$
- $w^2 \geq 0 \Rightarrow (w^0)^2 \geq |\vec{w}|^2 \Rightarrow |w^0| \geq |\vec{w}| \overset{w^0 \geq 0}{\Rightarrow} w^0 \geq |\vec{w}|$

Therefore,

$$v^0 w^0 \geq |\vec{v}||\vec{w}| \overset{CS}{\geq} |\vec{v} \cdot \vec{w}| \geq \vec{v} \cdot \vec{w} \Rightarrow v \cdot w = v^0 w^0 - \vec{v} \cdot \vec{w} \geq 0^2$$

(ii) One has that

$$v^2 = 0 \Rightarrow (v^0)^2 = |\vec{v}|^2$$

Thus,

$$v \cdot w = 0 \Rightarrow v^0 w^0 = \vec{v} \cdot \vec{w} \Rightarrow (w^0)^2 = \frac{|\vec{v} \cdot \vec{w}|^2}{(v^0)^2} \overset{CS}{\leq} \underbrace{\frac{|\vec{v}|^2}{(v^0)^2}}_{=1} |\vec{w}|^2 \Rightarrow w^2 = (w^0)^2 - |\vec{w}|^2 \leq 0$$

(iii) One has that

(I) $v^2 = 0 \Rightarrow (v^0)^2 = |\vec{v}|^2$

(II) $w^2 \geq 0 \Rightarrow (w^0)^2 \geq |\vec{w}|^2$

(III) $v \cdot w = 0 \Rightarrow v^0 w^0 = \vec{v} \cdot \vec{w}$

Therefore,

$$\text{(III)} \Rightarrow (v^0)^2 (w^0)^2 = |\vec{v} \cdot \vec{w}|^2 \overset{CS}{\leq} |\vec{v}|^2 |\vec{w}|^2 \overset{(I)}{\Rightarrow} (w^0)^2 \leq |\vec{w}|^2 \overset{(II)}{\Rightarrow} w^2 = (w^0)^2 - |\vec{w}|^2 = 0$$

Using (III) together with $w^2 = 0$ and (I),

$$|\vec{v} \cdot \vec{w}|^2 = (v^0)^2 (w^0)^2 = |\vec{v}|^2 |\vec{w}|^2 \Rightarrow \vec{v} \parallel \vec{w} \Leftrightarrow \exists \kappa \in \mathbb{R} : \vec{v} = \kappa \vec{w}$$

Using (III) and the above result,

$$v^0 w^0 = \vec{v} \cdot \vec{w} \Rightarrow v^0 w^0 = (\kappa \vec{w}) \cdot \vec{w} = \kappa |\vec{w}|^2 = \kappa (w^0)^2 \Rightarrow v^0 = \kappa w^0$$

Thus, one arrives at

$$v = \kappa w$$

□

Proof of Lemma 3. (i)
First, consider $\varphi_{(\nu)}(x, e)\Omega$

$$\left(\varphi_{(\nu)}(x, e)\Omega\right)(p) = \int_0^\nu ds_1 \cdots \int_0^\nu ds_n \left(\varphi^p\Big(x + (s_1 + \cdots + s_n)e\Big)\Omega\right)(p)$$

[2] CS means that we used the Cauchy-Schwarz inequality for the spatial part of the inner product,ie for $\vec{u}, \vec{v} \in \mathbb{R}^3$, $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$

Equation (2.20) for the point-local field φ^p yields

$$\left(\varphi^p\Big(x + (s_1 + \cdots + s_n)e\Big)\Omega\right)(p) = (2\pi)^{-\frac{3}{2}} e^{ip\cdot(x+(s_1+\cdots+s_n)e)} v^p(p) \tag{A.1}$$

and hence

$$\begin{aligned}\left(\varphi_{(\nu)}(x,e)\Omega\right)(p) &= (2\pi)^{-\frac{3}{2}} e^{ip\cdot x}\left(\int_0^\nu ds_1 \cdots \int_0^\nu ds_n e^{i(s_1+\cdots+s_n)p\cdot e}\right) v^p(p) = \\ &= (2\pi)^{-\frac{3}{2}} e^{ip\cdot x}\left(\int_0^\nu ds e^{is(p\cdot e)}\right)^n v^p(p) = \\ &= (2\pi)^{-\frac{3}{2}} e^{ip\cdot x}\frac{i^n}{(p\cdot e)^n}(1-e^{i\nu p\cdot e})^n v^p(p)\end{aligned}$$

Therefore,

$$v_\nu(p,e) = \frac{i^n}{(p\cdot e)^n}(1-e^{i\nu p\cdot e})^n\, v^p(p)$$

Inspecting the above equation and using (2.26), one easily verifies that it is a Wigner intertwiner for a lightlike string-local field (definition 5).

Now, lets show that $w_\nu(x-x',e,e')$ is well-defined. Let $f \in \mathcal{D}(\mathbb{R}^4)$, by (2.21) one has

$$w_\nu(f,e,e') \equiv \int d^4\xi f(\xi) w_\nu(\xi,e,e') = (2\pi)\int\limits_{H_m^+} d\mu_m(p)\check{f}(p)\frac{(-1)^n(1-e^{i\nu p\cdot e'})^n}{(p\cdot e)^n(p\cdot e')^n}M^P(p) \tag{A.2}$$

where M^p is the on-shell part of the two-point function of φ^p, which is a polynomial [8]. Now,

$$|w_\nu(f,e,e')| \leq 2\pi\int \frac{d^3\vec{p}}{2\omega_m(\vec{p})}\frac{|\check{f}(p)||M^P(p)|}{|\vec{e}|^n|\vec{e'}|^n}\frac{|1-e^{i\nu p\cdot e'}|^n}{(\omega_m(\vec{p})-\vec{p}\cdot\hat{e})^n(\omega_m(\vec{p})-\vec{p}\cdot\hat{e}')^n},$$

where $\hat{e} := \frac{\vec{e}}{|\vec{e}|}$ (analogous for $\hat{e}'$). Using that $|1-e^{i\nu p\cdot e'}|^n \leq 2^n$ and $\omega_m(\vec{p})-\vec{p}\cdot\hat{e}' \geq \omega_m(\vec{p})-|\vec{p}|$ (also for $\hat{e}$) we get

$$|w_\nu(f,e,e')| \leq (2\pi)\int \frac{d^3\vec{p}}{2\omega_m(\vec{p})}\frac{|\check{f}(p)||M^P(p)|}{|\vec{e}|^n|\vec{e'}|^n}\frac{2^n}{(\omega_m(\vec{p})-|\vec{p}|)^{2n}}$$

Considering spherical coordinates ($\kappa = |\vec{p}|, \vec{n}$) and using the fact that, there exists a $\kappa_0 > 0$ such that for all $\kappa > \kappa_0$

$$\omega_m(\kappa) - \kappa \geq \frac{m^3}{\kappa^2} \tag{A.3}$$

we have

$$|w_\nu(f,e,e')| \leq I_1 + I_2$$

where

$$\begin{aligned} I_1 &= 2\pi\int\limits_0^{\kappa_0}\kappa^2 d\kappa\int\limits_{S^2} d\Omega(\vec{n})\frac{1}{2\omega_m(\kappa)}\frac{|\check{f}(\kappa,\vec{n})||M^P(\kappa,\vec{n})|}{|\vec{e}|^n|\vec{e'}|^n}\frac{2^n}{(\omega_m(\kappa)-\kappa)^{2n}} \\ I_2 &= 2\pi\int\limits_{\kappa_0}^{\infty}\kappa^2 d\kappa\int\limits_{S^2} d\Omega(\vec{n})\frac{1}{2\omega_m(\kappa)}\frac{|\check{f}(\kappa,\vec{n})||M^P(\kappa,\vec{n})|}{|\vec{e}|^n|\vec{e'}|^n}\frac{2^n}{(\omega_m(\kappa)-\kappa)^{2n}}\end{aligned}$$

The first integral, I_1, is finite since $\check{f}$ is of rapid decrease and $\omega_m(\kappa) - \kappa > 0$. Using (A.3) and the bound $\omega_m(\kappa) \geq \kappa$, one sees that

$$I_2 \leq \pi \int_{\kappa_0}^{\infty} d\kappa \int_{S^2} d\Omega(\vec{n}) \frac{|\check{f}(\kappa, \vec{n})||M^P(\kappa, \vec{n})|}{|\vec{e}|^n |\vec{e'}|^n} \frac{2^n \kappa^{4n+1}}{m^{6n}}$$

which is also finite since $\check{f}$ is of rapid decrease. This show that $w_\nu(x-x', e, e')$ is well-defined.
(ii)
By the same considerations, one also shows that $\varphi(x, e)$ given by $v(p, e)$ is also a lightlike string-local field, and that $w(x - x', e, e')$ is also well-defined.

Now let's show that $w_\nu(x - x', e, e') \overset{\nu\to\infty}{\longrightarrow} w(x - x', e, e')$. Considering $w_\nu(x - x', e, e') - w(x - x', e, e')$, where

$$w(x - x', e, e') = (2\pi)^{-3} \int_{H_m^+} d\mu_m(p) e^{ip\cdot(x-x')} \frac{(-1)^n M^P(p)}{(p\cdot e)^n (p \cdot e')^n}$$

one has

$$w_\nu(f, e, e') - w(f, e, e') = (2\pi) \int_{H_m^+} d\mu_m(p) \check{f}(p)(-1^n) \frac{(1 - e^{i\nu p\cdot e'})^n - 1}{(p\cdot e)^n (p\cdot e')^n} M^P(p)$$

Lets show that the following integral,

$$\int_{H_m^+} d\mu_m(p) F(p) e^{i\nu p\cdot e'}, \text{ where } F(p) := \frac{\check{f}(p) M^P(p)}{(p\cdot e)^n (p \cdot e')^n}, \tag{A.4}$$

goes to zero, as ν goes to infinity. With this one can conclude the desired convergence. Pick a reference frame where $e' = (1, 0, 0, 1)$ and define $p_\pm := p^0 \pm p^3$, note that $p \cdot e' = p_+$. One can then write

$$d\mu_m(p) = \frac{dp_+ dp_1 dp_2}{p_+} \equiv \frac{dp_+ d^2\vec{\varrho}}{p_+}$$
$$p_- = \frac{m^2 + |\vec{\varrho}|}{p_+}$$

and

$$\int_0^\infty \frac{dp_+}{2p_+} \int d^2\vec{\varrho}\, F(\frac{m^2 + |\vec{\varrho}|}{p_+}, p_+, \vec{\varrho}) e^{i\nu p_+}$$

Writing $e^{i\nu p_+} = (i\nu)^{-1} \partial_{p_+} e^{i\nu p_+}$ and using the standard integration-by-parts trick (of Fourier analysis) n-times, one gets

$$\frac{1}{2i^n \nu^n} \int_0^\infty dp_+ \int d^2\vec{\varrho}\, \partial_{p_+}^n \left(\frac{F(\frac{m^2+|\vec{\varrho}|}{p_+}, p_+, \vec{\varrho})}{p_+} \right) e^{i\nu p_+}$$

Now, we want to show that $\left|\partial_{p_+}^n \left(\frac{F}{p_+}\right)\right| \leq M_n(p) \in L^1$. For this, lets show

$$\left| \frac{\partial_{p_+}^k F}{p_+^l} \right| \leq M_n'(p) \in L^1$$

Since $F \in \mathcal{S}(\mathbb{R}^4)$, for arbitrary $N \in \mathbb{N}$ and some $\kappa_0 > 0$

$$\left|\frac{\partial^k_{p_+} F}{p^l_+}\right| \leq \frac{c}{p^l_+}(1+p_-^2+p_+^2+|\vec{\varrho}|^2)^{-N} \leq c p_+^{-2N-l} \Rightarrow \frac{\partial^k_{p_+} F}{p^l_+} \in L^1 \text{for } p_+ > \kappa_0$$

Now, using that $p_- = \frac{m^2+|\vec{\varrho}|^2}{p_+}$, we get

$$\left|\frac{\partial^k_{p_+} F}{p^l_+}\right| \leq c p_+^{2N-l}(p_+^2+(m^2+|\vec{\varrho}|^2)^2+p_+^4+p_+^2|\vec{\varrho}|^2)^{-N} \leq c\ m^{-4N} p_+^{2N-l} \Rightarrow \frac{\partial^k_{p_+} F}{p^l_+} \in L^1 \text{for } p_+ < \kappa_0$$

This concludes the proof. □

Proof of Lemma 4. Lets divide the proof in 3 steps:

(i) Firtly, lets prove that $p \mapsto \tilde{h}(p)$ decreases rapidly on $V_+ \supseteq H_m^+$.

Consider $p \in \mathring{V}_+ \supseteq H_m^+$ $(m \neq 0)$ choose a basis $\{e^{(\mu)}\}_{\mu=1,2,3,4}$ such that $e^{(0)} = \frac{p}{|p|}$, where $|p| := \sqrt{p \cdot p}$.

$$\tilde{h}(p) = \int \frac{d^3\vec{e}}{2|\vec{e}|} h(|\vec{e}|, \vec{e}) e^{ip\cdot e} = \int \frac{d^3\vec{e}}{2|\vec{e}|} h(|\vec{e}|, \vec{e}) e^{i|p||\vec{e}|}$$

Using spherical coordinates $(\kappa, \vec{n})$ with $\kappa = |\vec{e}|$ and $\vec{n} \in S^2$,

$$\tilde{h}(p) = \frac{1}{2}\int_{S^2} d\Omega(\vec{n}) \int_0^\infty d\kappa f(\kappa, \vec{n}) e^{i|p|\kappa} \text{ with } f(\kappa, \vec{n}) := \kappa h(\kappa, \vec{n})$$

For $\lambda \in \mathbb{R}$, consider $\lambda \mapsto \tilde{h}(\lambda p)$.

$$\tilde{h}(\lambda p) = \frac{1}{2}\int_{S^2} d\Omega(\vec{n}) \int_0^\infty d\kappa f(\kappa, \vec{n}) e^{i\lambda|p|\kappa}$$

Using the indentity $e^{i\lambda|p|\kappa} = \frac{1}{i\lambda|p|}\partial_\kappa(e^{i\lambda|p|\kappa})$ together with integration-by-parts l times one gets

$$|\tilde{h}(\lambda p)| \leq \frac{1}{\lambda^l |p|^l} \underbrace{\left(\int_{S^2} d\Omega(\vec{n}) \int_0^\infty d\kappa |\partial^l_\kappa f(\kappa, \vec{n})|\right)}_{<\infty}$$

Hence, the limit

$$\lim_{\nu\to\infty} \int_0^\nu ds_1 \cdots \int_0^\nu ds_n \tilde{h}\left((s_1 + \cdots + s_n)p\right) \tag{A.5}$$

exists.

(ii) Consider the following family of functions $\left\{\frac{h(e)}{(p\cdot e+i\varepsilon)^n}\right\}_{\varepsilon\in\mathbb{R}_0^+}$, satisfying

- $\lim_{\varepsilon\to 0} \frac{h(e)}{(p\cdot e+i\varepsilon)^n} = \frac{h(e)}{(p\cdot e)^n}$

- $|\frac{h(e)}{(p\cdot e+i\varepsilon)^n}| \leq |\frac{h(e)}{(p\cdot e)^n}|$[3]

Applying Lebesgues theorem on dominated convergence, one has

$$t^n(p,h) := \int\limits_{H_0^+} d\sigma(e) h(e) \frac{i^n}{(p\cdot e)^n} = \lim_{\epsilon\to 0} \int\limits_{H_0^+} d\sigma(e) h(e) \frac{i^n}{(p\cdot e + i\epsilon)^n}$$

Noticing that

$$\frac{i^n}{(p\cdot e+i\epsilon)^n} = \left(\int\limits_0^\infty ds e^{is(p\cdot e+i\varepsilon)}\right)^n = \int\limits_0^\infty ds_1 \cdots \int\limits_0^\infty ds_n e^{i(s_1+\cdots+s_n)(p\cdot e+i\varepsilon)}$$

and using Fubinis theorem,

$$t^n(p,h) = \lim_{\epsilon\to 0} \int\limits_0^\infty ds_1 \cdots \int\limits_0^\infty ds_n e^{-(s_1+\cdots+s_n)\varepsilon} \underbrace{\left(\int\limits_{H_0^+} d\sigma(e) h(e) e^{i(s_1+\cdots+s_n)(p\cdot e)}\right)}_{\tilde{h}((s_1+\cdots+s_n)p)} \tag{A.6}$$

Considering now, the family of functions $\left\{e^{-(s_1+\ldots+s_n)\varepsilon}\tilde{h}\left((s_1+\cdots+s_n)p\right)\right\}_{\varepsilon\in\mathbb{R}_0^+}$ satisfiying

- $\lim\limits_{\varepsilon\to 0} e^{-(s_1+\ldots+s_n)\varepsilon}\tilde{h}\left((s_1+\cdots+s_n)p\right) = \tilde{h}\left((s_1+\cdots+s_n)p\right)$
- $|e^{-(s_1+\ldots+s_n)\varepsilon}\tilde{h}\left((s_1+\cdots+s_n)p\right)| \leq |\tilde{h}\left((s_1+\cdots+s_n)p\right)|$

and using again Lebesgues theorem, one has

$$\begin{aligned} t^n(p,h) &= \int\limits_0^\infty ds_1 \cdots \int\limits_0^\infty ds_n \lim_{\varepsilon\to 0} e^{-(s_1+\ldots+s_n)\varepsilon}\tilde{h}\left((s_1+\cdots+s_n)p\right) = \\ &= \int\limits_0^\infty ds_1 \cdots \int\limits_0^\infty ds_n \tilde{h}\left((s_1+\cdots+s_n)p\right) \end{aligned}$$

(iii) Having proved that

$$t^n(p,h) = \lim_{\nu\to\infty} t^n_{(\nu)}(p,h)\ ,\ \text{where } t^n_{(\nu)}(p,h) := \int\limits_0^\nu ds_1 \cdots \int\limits_0^\nu ds_n \tilde{h}\left((s_1+\cdots+s_n)p\right) \tag{A.7}$$

we now determine a bound for it. Performing the change of variables $s_i \to p_0 s_i$, for $i = 1, ..., n$,

$$t^n_{(\nu)}(p,h) = \frac{1}{p_0^n} \int\limits_0^{p_0\nu} ds_1 \cdots \int\limits_0^{p_0\nu} ds_n \tilde{h}\left((s_1+\cdots+s_n)\frac{p}{p_0}\right)$$

[3] The dominating function $|\frac{h(e)}{(p\cdot e)^n}|$ is integrable on H_0^+ since $m \neq 0$

Using that $\tilde{h}$ decreases rapidly in V_+ we have that

$$|t^n_{(\nu)}(p,h)| \leq \frac{C_N}{p_0^n} \int\limits_0^{p_0\nu} ds_1 \cdots \int\limits_0^{p_0\nu} ds_n \left[1 + (s_1 + ... + s_n) \left\| \frac{p}{p_0} \right\|_1 \right]^{-N}$$

Since $\left\|\frac{p}{p_0}\right\|_1 := \underbrace{\left|\frac{p_0}{p_0}\right|}_{=1} + \sum\limits_{i=1}^{3} \left|\frac{p_i}{p_0}\right| \geq 1$

$$|t^n_{(\nu)}(p,h)| \leq \frac{K_\nu}{p_0^n}$$

where

$$K_\nu = C_N \int\limits_0^{p_0\nu} ds_1 \cdots \int\limits_0^{p_0\nu} ds_n \left[1 + (s_1 + ... + s_n)\right]^{-N} < \infty$$

Then,

$$|t^n(p,h)| = \underset{\nu\to\infty}{lim} |t^n_{(\nu)}(p,h)| \leq \underset{\nu\to\infty}{lim} \frac{K_\nu}{p_0^n} = \frac{K}{p_0^n}$$

where $K = \underset{\nu\to\infty}{lim} K_\nu < \infty$.

$\square$

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